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Existence of Random Attractors for a Stochastic Strongly Damped Plate Equations with Multiplicative Noise

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Original Research Article

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Abstract

In this article, we study the asymptotic dynamics of a stochastic strongly damped plate system with homogeneous Neumann boundary conditions and multiplicative noise. First, we investigate the existence and uniqueness of solutions in infinite-dimensional dynamical systems using the notion of mild solutions, and then we examine the presence of a bounded absorbing set. Finally, we investigate the asymptotic compactness by using the decomposition technique to prove the existence of a random attractor.

Keywords: Plate equations; random attractors; strongly damped; dynamical systems.

2020 Mathematics Subject Classification: 37L55, 35R60, 35B40, 35B41, 35B45.

1 Introduction

Consider the stochastic strongly damped plate equation with multiplicative noise in a bounded, open set Ω of $\mathbb{R}^n(n=5)$ with smooth boundary $\partial\Omega$:

$$\begin{cases} u_{tt}+\alpha\Delta^2 u_t+\Delta^2 u+\varepsilon u+g(u)=f(x)+cu\circ\dfrac{dW(x,t)}{dt}, & x\in\Omega, t\geq 0,\\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega, t\leq 0,\\ u|_{\partial\Omega}=\dfrac{\partial u}{\partial n}|_{\partial\Omega}=0, & t\geq 0, \end{cases} \tag{1.1}$$

where ε,α and c are positive constants, $\triangle$ represents the Laplacian with respect to $x\in\mathbb{R}^5$, $u=u(x,t)$ is a real function in $\Omega\times[0,+\infty)$, where $u_0\in H_0^2(\Omega)$, $u_1\in L^2(\Omega)$, $f(x)\in H_0^1(\Omega)\cap H^2(\Omega)$ are represents given external forces. $W(x,t)$ is an independent two-sided real-valued Wiener process on the probability space $(\Omega,\mathcal{F},\mathbb{P})$,where

$$\Omega=\{\omega=(\omega_1,\ \omega_2,.....,\ \omega_m)\in C(\mathbb{R},\mathbb{R}^m):\omega(0)=0\},$$

is endowed with the compact open topology and $\mathbb{P}$ is its corresponding Wiener measure. $\mathcal{F}$ is the completion of the Borel σ-algebra with respect to $\mathbb{P}-$ on Ω. We identify $W(t)$ with $(W_1(t),\ W_2(t),.....,\ W_m(t))$, i.e.,

$$W(t)=(W_1(t),\ W_2(t),.....,\ W_m(t)),\ t\in\mathbb{R}.$$

Define a time shift. $(\theta_t)_{t\in\mathbb{R}}$ on Ω by

$$\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t),\quad t\in\mathbb{R},\ \omega\in\Omega.$$

The nonlinear term g is a C^1- function with $g(0)=0$ and satisfies the following conditions:

(h_1) There exists constants $0\leq p\leq 4, n\geq 5$, and a positive constant C_1 such that

$$|g'(u)|\leq C_1(1+|u|^p)\ ,\forall\ u\in\mathbb{R}, \tag{1.2}$$

and
(h_2) There exists positive constants

$$\begin{cases} \lim\inf\limits_{|u|\to\infty}\dfrac{g(u)}{u}\geq 0\ ,\forall\ u\in\ \mathbb{R},\\ \lim\inf\limits_{|u|\to\infty}\dfrac{g(u)u-\mu_i G(u)}{u^2}\geq 0,\ \forall\ u\in\mathbb{R} \end{cases} \tag{1.3}$$

(h_3) There exists constants $k>0$ and μ_1 such that for all$\mu\in(0,\mu_1)$, there is a value of $\mu_i\in\mathbb{R}$ that satisfies.

$$\begin{cases} kG(u)-\mu u^2+c_\mu\leq ug(u),\ \forall\ u\in\mathbb{R}\\ G(u)\geq\mu\|u\|^{P+2}+c_\mu\|u\|^2,\ \forall\ u\in\mathbb{R} \end{cases} \tag{1.4}$$

where $G(s) = \int_0^s g(r)dr$.

The study of asymptotic behavior of dynamical systems is a crucial issue in mathematical physics, with notable progress made in recent years. In deterministic systems, the global attractor, a compact set that is invariant and attracts nearby points, is central to understanding dynamics (as seen in Temam [1]. This paper focuses on the random attractors of equation (1.1) when the forcing term is time-independent. To study the equation's dynamics, two parametric spaces are introduced: one for deterministic forcing and the other for stochastic perturbations. The existence and upper semi-continuity of the global attractor and pullback attractor (or kernel sections) for deterministic autonomous and non-autonomous systems have been widely studied in relation to this problem(as seen in references [2, 3, 4]).

Several authors have introduced a distinct notion of attractors for stochastic partial differential equations, including H. Crauel [5, 6], Morimoto [7], L. Arnold [8, 9], J. Duan, K. Lu and B. Schmalfuß [10], J. Hale, X. Lin and G. Raugel [4], T. Caraballo, and J. Langa [11, 12]. They have studied the existence and upper semi-continuity of attractors for deterministic and random dynamical systems, respectively. They established general criteria for the existence and upper semi-continuity of attractors in non-autonomous stochastic evolution equations with time-dependent external terms and multiplicative noise. Wang[13] developed a useful theory on the existence and upper semi-continuity of random attractors by introducing two parametric spaces and applying it to non-autonomous stochastic reaction-diffusion equations and wave equations. For further information (see [14, 15]).

In recent years, numerous advancements have been made in the study of systems related to equation (1.1). The dynamics of deterministic hyperbolic equations have been explored and shown to have global attractors, which are finite-dimensional objects despite being subsets of an infinite-dimensional phase space. Some examples include the existence of global attractors for linear damped plate equations with critical exponent (A. Khanmamedov[16], G. Yue and C. Zhong [17]), nonlinear damped plate equations [18, 19], strongly damped plate equations with white noise (Ma et al.[20]), and strongly damped wave equations ([21, 22, 23, 24, 25]). Further references for this area of study can be found (see[26, 27, 28, 29, 30]). In [31], the analysis of fractional-order proportional delay physical models was studied via a novel transform. Bhadane P. et al. in [32] investigated the approximate solution of the fractional Black-Scholes European option pricing equation by using ETHPM. Hamoud A. [33, 34] provides recent advances on reliable methods for solving Volterra-Fredholm integral and integro-differential equations, and discusses some powerful techniques for solving nonlinear Volterra-Fredholm integral equations.

Recently, researchers have discovered the presence of random attractors for various equations, as indicated in references([35, 36, 37, 38, 39]). However, there is a lack of research on random attractors for equation (1.1). This article aims to study the existence of random attractors for the system (1.1) - (1.2). Proving compactness of the generated random dynamical system is challenging, but its asymptotic compactness can be established by using the solution decomposition method, as shown in references(see[28, 40, 41]).

The paper is structured as follows. Section 2 reviews basic concepts and properties of general random dynamical systems. Section 3 establishes the framework for (1.1) by providing the basic settings, demonstrating that it generates a random dynamical system in an appropriate function space, and establishing the existence and uniqueness of solutions. In Section 4, uniform energy estimates for the solutions of (1.1) defined on $\mathbb{R}^5$ are derived with the aim of proving the existence of a bounded random absorbing set and the asymptotic compactness of the associated random dynamical system as $t \to \infty$. In section 5, we discuss the decomposition of solutions in order to obtain the asymptotic compactness. Then, existence of a random attractor is proven in the Section 6. Finally, we give the conclusion.

2 Random Dynamical Systems

This section serves to refresh our understanding of basic concepts related to RDS and random attractors (further details can be found in [5, 6]) in order to obtain our main results, it's crucial to recall some definitions and properties concerning the asymptotic behavior of random dynamical system defined by (1.1). We consider $(X, \|\cdot\|_X)$ to be a separable Hilbert space with Borel $\sigma-$algebra $\mathfrak{B}(X)$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ to be a metric dynamical system.

Definition 2.1. *[5] Consider a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$. The mapping $\phi : \mathbb{R}^+ \times \Omega \times X \to X$ is defined as a RDS if it is measurable with respect to the sigma algebra $(\mathcal{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathcal{B}(X), \mathcal{B}(X))-$ and satisfies the following properties:*
(i) $\phi(0, \omega)x = x$;
(ii) $\phi(s, \theta_t\omega) \circ \phi(t, \omega)x = \phi(s + t, \omega)x$;
for all $s, t \in \mathbb{R}^+$, $x \in X$ and $\omega \in \Omega$.
If, in addition, ϕ is continuous with respect to $t \leq 0$ and $\omega \in \Omega$,it is referred to as a continuous RDS.

Definition 2.2. *[6] A mapping $\Phi(t, \tau, \omega, x) : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is referred to as a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$,if it satisfies the following conditions for all $\tau \in \mathbb{R}$,$\omega \in \Omega$ and $t, s \in \mathbb{R}^+$:*
i) $\Phi(t, \tau, \omega, x) : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}, \mathcal{B}(\mathbb{R}))$ is a measurable mapping with respect to the sigma algebra,
ii) $\Phi(0, \tau, \omega, x)$ is the identity function on X,
iii) $\Phi(t + s, \tau, \omega, x) = \Phi(t, \tau + s, \theta_s\omega, x) \circ \Phi(s, \tau, \omega, x)$,
iv) $\Phi(t, \tau, \omega, x) : X \to X$ is continuous.

Definition 2.3. *[42] A set-valued mapping $B : \Omega \to 2^X$ is referred to as a random closed set if for all $B(\omega)$ is a closed set,non-empty, and The function$\omega \mapsto d(x, B(\omega))$ is measurable for all $x \in X$, $\omega \in \Omega$. A random set $\mathcal{B} := \{B(\omega)\}_{\omega\in\Omega}$ is referred to as tempered if.*

$$\lim_{t\to\infty} e^{-\eta t} d(B(\theta_{-t}\omega)) = 0,$$

for a.e. $\omega \in \Omega$ and all $\eta > 0$, where $d(B) := \sup_{x,y\in B} d(x, y)$.

Definition 2.4. *[43] Let $\mathcal{D}$ be a collection of random subset of X and $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, then K is called an absorbing set of $\Phi \in \mathcal{D}$, This means that for any $\tau \in \mathbb{R}, \omega \in \Omega$ and $B \in \mathcal{D}$, there exists a $T = T(\tau, \omega, B) > 0$ such that for all $B \in \mathcal{D}$, the following holds:*

$$\Phi(t , \tau, \theta_{-t}\omega, B(\tau, \theta_{-t}\omega)) \subseteq K(\tau, \omega) \ , \ \forall \ t \ \geq T.$$

Definition 2.5. *[44] Let $\mathcal{D}$ be the collection of all tempered random sets in X, and a random set $\mathcal{A} := A\{(\omega)\}_{\omega\in\Omega} \in X$ is called a random attractor for the RDS ϕ if P-a.s.*
(i) $\mathcal{A}$ is a random compact set, i.e. $A(\omega)$ is nonempty and compact for a.e. $\omega \in \Omega$ and $\omega \mapsto d(x, A(\omega))$ is measurable for every $x \in X$;
(ii) $\mathcal{A}$ is $\phi-$invariant, i.e. $\phi(t, \omega, A(\omega)) = A(\theta_t\omega)$, for all $t \geq 0$ and a.e. $\omega \in \Omega$;
(iii) $\mathcal{A}$ attracts every set in X, i.e. for all bounded (and non-random) $B \subset X$,

$$\lim_{t\to\infty} dist(\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), A(\omega)) = 0, \quad a.e.\ \omega \in \Omega.$$

Lemma 2.1. *[6] Suppose there exists a random compact set $\{K(\omega)\}_{\omega\in\Omega}$ that can absorb all bounded, non-random sets $B \in \mathcal{D}$, for the continuous random dynamical system on E over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$. Then, the set*

$$\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega} \ = \overline{\cup_{B\subset X}\Lambda_B(\omega)},$$

is a global attractors for ϕ, where the union is taken over all bounded $B \subset X$, and $\Lambda B(\omega)$ is the $\omega-$ limits set of B, defined as:

$$\Lambda_B(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} (\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), A(\omega))}, \ \omega \in \Omega.$$

3 Existence and Uniqueness of Solutions

In this section, we explore the existence and uniqueness of solutions to system (1.1) in a bounded subset of $\Omega \subset \mathbb{R}^n, (n = 5)$. We make the following assumptions: $(i) - (iii)$ are satisfied, the space E and the probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ are defined as in Section 1. Further, let the set $A = \Delta^2$ is defined as the collection of all functions satisfies Neumann boundary condition on Ω. Then, domain of A is defined as $D(A) = \{u \in H^4(\Omega) \cap H_0^2(\Omega) : \frac{\partial u}{\partial n}|_{\partial\Omega} = 0\}$. Clearly, A is a self-adjoint and positive linear operator with eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots, \lambda_i \to +\infty \quad (i \to +\infty).$$

Let $E = H_0^2(\Omega) \times L^2(\Omega)$, which is a separable Hilbert space endowed with the usual norm

$$\|Y\|_{H_0^2 \times L^2} = (\|\Delta u\|^2 + \|v\|^2)^{\frac{1}{2}} \quad \text{for} \quad Y = (u, v)^\top, \tag{3.1}$$

where $\|\cdot\|$ denotes the usual norm in $L^2(\Omega)$ and $\top$ stands for the transposition. it could be defined the powers A^r of A for $r \in \mathbb{R}$. The space $V_{2r} = D(A^r)$ is the Hilbert space with the standard inner product and norm, respectively

$$((\cdot,\cdot))_{D(A^r)} = (A^r\cdot, A^r\cdot), \quad \|\cdot\|_{D(A^r)} = \|A^r\cdot\|, ((u,u)) = \int_\Omega \Delta u \Delta v dx, \quad \|\Delta u\| = ((u,u))^{\frac{1}{2}}.$$

Let $\forall u, v \in H_0^2(\Omega)$. Especially, (u, v) and $\|\cdot\|$ denote the $L^2(\Omega)$ inner product and norm respectively, $(u, u) = \int_\Omega uvdx$, $\|u\| = (u, u)^{\frac{1}{2}}$ $\forall u, v \in L^2(\Omega)$. Therefore, the injection from $D(A^r) \hookrightarrow D(A^s)$ is compact if $r > s$. This leads to the satisfaction of the generalized Poincaré inequality

$$\|u\|_r^2 \geq \lambda_0 \|u\|_s^2 \quad \text{Where} \quad \lambda_0 > 0 \quad \text{is the first eigenvalue of} \quad A.$$

The goal is to turn problem (1.1) into a deterministic system with random parameters and no noise terms and demonstrate that it creates a random dynamical system. This is accomplished by using the Ornstein-Uhlenbeck process derived from Brownian motion, which follows *Itô* differential equation

$$dz + \alpha z dt = dW(t), \tag{3.2}$$

therefore, the solution is given as follows:

$$\begin{aligned} \theta_t\omega(s) &= \omega(t+s) - \omega(t), \\ z(\theta_t\omega) &= -\alpha \int_\infty^0 e^s(\theta_t\omega)(s)ds, \quad s, t \in \mathbb{R}, \quad \omega \in \Omega. \end{aligned} \tag{3.3}$$

It has been established in [9, 28, 45] that the random variable $|z(\omega)|$ is tempered, and there exists a set $\bar{\Omega} \subseteq \Omega$, which is θ_t -invariant and has full measure according to P, such that for all $\omega \in \bar{\Omega}$, the mapping $t \mapsto z(\theta_t\omega)$ is continuous with respect to t

$$\lim_{t\to\infty} e^{-\alpha t}|z(\theta_{-t}\omega)| = 0, \ \ \forall \ \alpha > 0, \ \omega \in \bar{\Omega}. \tag{3.4}$$

Equation (3.3) has a random fixed point in the context of random dynamical systems, resulting in a stationary solution called the stationary Ornstein-Uhlenbeck process (refer to [5, 6, 28, 35] for further information). For ease of use, in the following, it is denoted as Ω instead of $\bar{\Omega}$.

Lemma 3.1. *(Refer to [29, 40, 45]) The Ornstein-Uhlenbeck process in equation 3.3, denoted as $z(\theta_t\omega)$, is* *rewritten as:*

$$\begin{cases} \lim\limits_{t\to\pm\infty} \dfrac{|z(\theta_t\omega)|}{|t|} = 0, \\ \lim\limits_{t\to\pm\infty} \dfrac{1}{t}\displaystyle\int_{-t}^{0} z(\theta_s\omega)ds = E[z(\theta_s\omega)] = 0, \\ \lim\limits_{t\to\pm\infty} \dfrac{1}{t}\displaystyle\int_{-t}^{0} z(\theta_s\omega)ds = E[z(\theta_s\omega)] = \dfrac{1}{\sqrt{\pi\delta}}, \\ \lim\limits_{t\to\pm\infty} \dfrac{1}{t}\displaystyle\int_{-t}^{0} |z(\theta_s\omega)|^2 ds = E[|z(\theta_s\omega)|^2] = \dfrac{1}{2\delta}, \end{cases} \tag{3.5}$$

by (3.5), there exists $\mathrm{T}_1(\omega) > 0$ *such that for all* $t \geq \mathrm{T}_1(\omega)$*,*

$$\int_{-t}^{0} z(\theta_s\omega)ds < \frac{2}{\sqrt{\pi\delta}}t, \quad \int_{-t}^{0} |z(\theta_s\omega)|^2 ds < \frac{1}{2\delta}t. \tag{3.6}$$

To make equation (1.1) easier to evaluate, it is useful to convert it into a first-order equation in time.$v = u_t + \varepsilon u - cuz(\theta_t\omega)$*. This can be achieved by defining*

$$\begin{cases} \dfrac{du}{dt} = v - \varepsilon u + cuz(\theta_t\omega), \\ \dfrac{dv}{dt} = (\varepsilon - \alpha A)v - (\varepsilon - \alpha A + A + \mu)u - g(u), \\ \qquad - (v - 2\varepsilon u + cuz(\theta_t\omega) + (A-1)\alpha u)z(\theta_t\omega) + f(x), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) = u_1(x) + \varepsilon u_0(x) - cu_0(x)z(\theta_t\omega), \end{cases} \tag{3.7}$$

Let

$$Y = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad L = \begin{pmatrix} \varepsilon I & -I \\ \varepsilon I - \alpha A + A + \mu & -\varepsilon I + \alpha A \end{pmatrix},$$

and

$$Q(t,\omega,Y) = \begin{pmatrix} cuz(\theta_t\omega) \\ -g(u) - (v - 2\varepsilon u + cuz(\theta_t\omega) + (A-1)\alpha u)z(\theta_t\omega) + f(x) \end{pmatrix},$$

Then, equation (3.7) has the simple matrix form

$$Y' + LY = Q(t,\omega,Y) \tag{3.8}$$

it is defined

$$\psi_1 = u, \quad \psi_2 = \frac{du}{dt} + \varepsilon u - cuz(\theta_t\omega), \tag{3.9}$$

given a positive constant ε*, equation (3.7), can be expressed as an equivalent system with random coefficients in* E *as follows:*

$$\begin{cases} \dfrac{d\psi_1}{dt} = \psi_2 - \varepsilon\psi_1 + c\psi_1 z(\theta_t\omega), \\ \dfrac{d\psi_2}{dt} = (\varepsilon - \alpha A)\psi_2 - (\varepsilon - \alpha A + A + \mu)\psi_1 - g(\psi_1), \\ \qquad - (\psi_2 - 2\varepsilon\psi_1 + c\psi_1 z(\theta_t\omega) + (A-1)\alpha\psi_1)z(\theta_t\omega) + f(x), \\ \psi_1(x,0) = u_0(x), \quad \psi_2(x,0) = v_0(x) = u_1(x) + \varepsilon u_0(x) - cu_0(x)z(\theta_t\omega), \end{cases} \tag{3.10}$$

equation (3.10), the random differential equation, can be expressed in vector form as follows:

$$\begin{cases} \psi\prime + L\psi = Q(\psi, t, \omega), \\ \psi_0 = (\psi_1(x,0), \psi_2(x,0)) = (u_0(x), u_1(x) + \varepsilon u_0(x) - cu_0(x)z(\theta_t\omega))^{\top}, \end{cases} \tag{3.11}$$

whereas

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad L = \begin{pmatrix} \varepsilon I & -I \\ \varepsilon I - \alpha A + A + \mu & -\varepsilon I + \alpha A \end{pmatrix},$$

and

$$Q(\psi, t, \omega) = \begin{pmatrix} c\psi_1 z(\theta_t\omega) \\ -g(\psi_1) - (\psi_2 - 2\varepsilon\psi_1 + c\psi_1 z(\theta_t\omega) + (A-1)\alpha\psi_1)z(\theta_t\omega) + f(x) \end{pmatrix}.$$

In accordance with references [1, 42, 46], it is established that the operator L in equation(3.11) is the infinitesimal generator of C_0-semigroup e^{Lt}, of contractions on E for $t > 0$, and also generates a C_0-semigroup e^{-Lt} of contractions on E. Due to assumptions (h_2) and the embedding relation $H_0^2(\Omega) \hookrightarrow L^{10}(\Omega)$, it can be verified that $Q(\psi, t, \omega) : E \to E$ is locally Lipschitz continuous with respect to φ for each $\omega \in \Omega$, using the classical semigroup theory for (local) existence and uniqueness solution of evolution differential equation [46]. This leads to the following theorem.

Theorem 3.2. *Assume that $h_1 - h_3$ hold, for each $\omega \in \Omega$ and for any $\psi_0 \in E$, there exists $T > 0$ such that (3.11) has a unique mild function $\psi(\cdot, \omega, \psi_0) \in C([0, +\infty); E)$ such that $\psi(0, \omega, \psi_0) = \psi(0)$ satisfies the integral equation*

$$\psi(t, \omega, \psi_0) = e^{-Lt}\psi_0(\omega) + \int_0^t e^{L(t-s)} Q(\psi(s, \omega, \psi_0), \theta_s\omega, s)ds. \tag{3.12}$$

However, $\psi(t, \omega, \psi_0)$ is jointly continuous in (ψ_0) and measurable in ω.

According to Theorem 3.1, it is known that for P-a.s. each $\omega \in \Omega$, , the following results hold for all $\mathrm{T} > 0$
(i)- if $\psi_0(\omega) \in E$ then, $\psi(\cdot, \omega, \psi_0) \in C([0, +\infty; E)$,
(ii)- $\psi(t, \omega, \psi_0)$ is jointly continuous into t and measurable in $\psi_0(\omega)$,
(iii)- the solution mapping of equation(3.11) possesses the properties of a Random Dynamical System.

The solution $\psi(\cdot, \omega, \psi_0)$ of equation(3.11) defines a continuous random dynamical system over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$, and this solution mapping has been noticed to be unique.

$$\begin{aligned} &\bar{\Phi}(t, \omega) : \mathbb{R} \times \Omega \times E \mapsto E, t \geq 0, \\ &\psi(0, \omega) = (u_0(\omega), v_0(\omega),)^\top \mapsto (u(t, \omega), v(t, \omega),)^\top = \psi(t, \omega), \end{aligned} \tag{3.13}$$

generates a random dynamical system and, in addition,

$$\Phi(t, \omega) : Y_0 = \psi(0, \omega) + (0, cuz(\theta_0\omega))^\top \mapsto Y(t, \omega, Y_0) = \psi(t, \omega, \psi_0) + (0, cuz(\theta_t\omega))^\top, \tag{3.14}$$

where $Y_0 = (u_0, u_1)^\top$ and $\psi_0 = (u_0, u_1 + cuz(\theta_t\omega))^\top$, $\Phi(t, \omega)$ is a continuous random dynamical system associated with the problem (3.8) on E. $\Phi(t, \omega)$ has a relationship with $\bar{\Phi}(t, \omega)$

$$\bar{\Phi}(t, \omega) = R(\theta_t\omega)\Phi(t, \omega)R^{-1}(\theta_t\omega) \tag{3.15}$$

The transformation $R(\theta_t\omega) : (a, b)^\top \mapsto (a, b - cuz(\theta_t\omega)^\top$ is a homeomorphism of E, and it is also defined

$$\varphi_1 = u = \psi_1, \ \varphi_2 = u_t + \varepsilon u, \tag{3.16}$$

similar to equation(3.11), it was obtained that.

$$\begin{cases} \varphi' + H(\varphi) = Q_\varepsilon(\varphi, t, \omega) \\ \varphi_0(x, 0) = (u_0, v_0)^\top = (u_0(x), u_1(x) + \varepsilon u_0(x))^\top, \end{cases} \tag{3.17}$$

whereas

$$\varphi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad H(\varphi) = \begin{pmatrix} v - \varepsilon u \\ (\varepsilon - \alpha A + A + \mu)u - (\varepsilon - \alpha A)v \end{pmatrix},$$

and

$$Q_\varepsilon(\varphi, \omega, t) = \begin{pmatrix} 0 \\ cuz(\theta_t\omega) - g(u) + f(x) \end{pmatrix}.$$

An isomorphism $T_\varepsilon\varphi = (\varphi_1, \varphi_2 - \varepsilon\varphi_1)^\top, \quad \varphi = (\varphi_1, \varphi_2)^\top \in E$, *was introduced. It has an inverse isomorphism* $T_{-\epsilon}\varphi = (\varphi_1, \varphi_2 + \varepsilon\varphi_1)^\top$, *and it follows that* (θ, φ) *maps.*

$$\bar{\Phi}_\varepsilon(t,\omega) = T_\varepsilon\bar{\Phi}(t,\omega)T_{-\varepsilon} : \varphi_0 \mapsto \varphi(t,\omega,\varphi_0) \tag{3.18}$$

A random dynamical system associated with (3.16) is defined, where $\varphi_0 = (u_0, u_1 + \varepsilon u_0 - cu_0 z(\theta_0\omega))^\top$ *and* $T_\varepsilon : (a,b)^\top \mapsto (a, b+\varepsilon a)^\top$ *is an isomorphism of E. It should be noted that all the random dynamical systems* $\Phi(t,\omega), \bar{\Phi}(t,\omega), \bar{\Phi}_\varepsilon(t,\omega)$ *are equivalent. This article will study the existence of random attractor for RDS* Φ *based on this theorem.*

4 Uniform Estimates of Solutions

This section will demonstrate the existence of a random absorbing set for the RDS $\varphi(t,\omega,\varphi_0(\omega)), t \geq 0$ in the space E, and provide uniform estimaties on the solutions of (3.11) defined on $\mathbb{R}^n$ (n=5). For this purpose, a new Hilbert space E will be introduced. It is defined as $(\varphi,\tilde{\varphi})_E = \gamma(A^{\frac{1}{2}}u_1, A^{\frac{1}{2}}u_2) + (v_1 v_2)$ and $\|\varphi\|_E = (\varphi,\varphi)_E^{\frac{1}{2}}$ for any $\varphi = (u_1,v_1)^\top, \ \tilde{\varphi} = (u_2,v_2)^\top \in E$, where γ is chosen

$$\gamma = \frac{4+\alpha\lambda_1+\beta_1}{4+2(\alpha\lambda_1+\beta_1)\alpha+\beta_2^2/\lambda_1}, \tag{4.1}$$

It is clear that the norm $\|\cdot\|_E$ is equivalent to the usual norm $\|\cdot\|_{H_0^2\times L^2}$ of E.

Lemma 4.1. *For any* $\varphi = (u,v)^T \in E$, *it follows that*

$$(H(\varphi),\varphi)_E \geq \frac{\varepsilon}{2}\|\varphi\|_E^2 + \frac{\varepsilon}{4}\|u\|_2^2 + \frac{\alpha}{2}\|v\|^2.$$

Proof. Let $\varphi(t) = (u(t),v(t))^T$ and $H(\varphi) \in E$, it is obtained

$$\begin{aligned}(H(\varphi),\varphi)_E =& \|\varphi\|_E^2 - \varepsilon\|u\|_2^2 + (\alpha-\varepsilon)\|v\|^2 - \varepsilon(\alpha-\varepsilon)(u,v)\\ \geq& \varepsilon\|u\|_2^2 + (\alpha-\varepsilon)\|v\|^2 - \varepsilon(\alpha-\varepsilon)(u,v)\\ =& \frac{\varepsilon}{2}\|\varphi\|_E^2 + \varepsilon\|u\|_2^2 + \frac{\alpha}{2}\|v\|^2.\end{aligned}$$

□

Lemma 4.2. *Under the assumptions* $(h_1)-(h_3)$, *there exists a random variable* $r_1(\omega) > 0$ *and a bounded ball* $B_0(t,\omega) \subset E$, *centered at 0 with random radius* $r_0(\omega) > 0$, $B_E(0, r_0(\omega) \in \mathcal{D}(E)$, *For any bounded non-random set* $B \subset \mathcal{D}(E)$, *there exists a deterministic time* $T = T(t,\omega,B) \geq 0$, *such that the solution* $\varphi(t,\omega;\varphi(\omega))$ *of equation(3.17) with initial value* $(u_0, u_1+\varepsilon u_0, \eta_0)^T \in B$ *satisfies, for almost all with respect to* $P-a.s.\ \omega \in \Omega$,

$$\|\varphi(t,\omega;\varphi(0,\omega))\|_E \leq r_0^2(\omega), \ t \geq T(B).$$

Proof. For any $\omega \in \Omega, \quad t \geq 0$, let $\varphi(t) = (u(t),v(t)) \in E$ be a mild solution of (3.17). By taking the inner product $(\cdot,\cdot)_E$ of (3.17) with $\varphi(t) = (u,v) = (u, u_t + \varepsilon u - cuz(\theta_t\omega))^\top$, it is obtained that

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_E^2 + (H(\varphi,\varphi))_E = (Q(\varphi,\omega,t),\varphi), \tag{4.2}$$

As a result of Lemma 4.1,

$$(H(\varphi,\varphi))_E = \frac{\varepsilon}{2}\|\varphi\|^2 + \varepsilon\|u\|_2^2 + \frac{\alpha}{2}\|v\|^2, \tag{4.3}$$

let us evaluate the right side of equation (4.2)

$$(Q(\varphi,\omega,t),\varphi) = ((cuz(\theta_t\omega),u)) + (cu(\varepsilon-\alpha A)z(\theta_t\omega),w) + (c^2uz^2(\theta_t\omega),w) \\ +(cvz(\theta_t\omega),w) - (g(u),w) + (f(x),w) \tag{4.4}$$

By using the Cauchy-Schwartz inequality, it can be determined that

$$((cuz(\theta_t\omega),u)) \le |c||z(\theta_t\omega)|\|u\|_2^2, \tag{4.5}$$

$$\varepsilon(cuz(\theta_t\omega),w) \le \varepsilon|c||z(\theta_t\omega)|\|u\|\|w\| \le \frac{\varepsilon|c||z(\theta_t\omega)|}{2\sqrt{\lambda_0}}(\|u\|_2^2+\|w\|^2), \tag{4.6}$$

$$(c^2uz^2(\theta_t\omega),w) \le |c|^2|z(\theta_t\omega)|^2\|u\|\|w\| \le \frac{c^2}{2\sqrt{\lambda_0}}|z(\theta_t\omega)|^2(\|u\|_2^2+\|w\|^2), \tag{4.7}$$

$$(cwz(\theta_t\omega),w) \le |c||z(\theta_t\omega)|\|w\|^2, \tag{4.8}$$

$$(f(x),w) \le \frac{2}{\alpha}\|f(x)\|^2 + \frac{\alpha}{8}\|w\|^2, \tag{4.9}$$

$$\alpha(c\Delta uz(\theta_t\omega),\Delta w) \le \alpha|c||z(\theta_t\omega)|\|u\|\|w\| \le \frac{\alpha\sqrt{\lambda_0}|c||z(\theta_t\omega)|}{2}(\|u\|_2^2+\|w\|^2), \tag{4.10}$$

By using $(h_2),(h_3)$ and the and the Hölder inequality, the nonlinear term in (4.4) can be estimated as follows:

$$\begin{aligned}(g(u),w) &= (g(u),u_t+\varepsilon u - cuz(\theta_t\omega)) \\ &= \tfrac{d}{dt}\textstyle\int_U G(u)dx + \varepsilon\,(g(u),u) - cuz(\theta_t\omega)(g(u),u)\,.\end{aligned} \tag{4.11}$$

As a result of (3.3), (h_1), *and* (h_2), and thepoincarè inequality, there exists positive constants μ_1,μ_2

$$(g(u),u) - k\tilde{G}(u) + \mu_1\|u\|_2^2 + \mu_2 \ge 0, \tag{4.12}$$

It is deduced from (1.4) that, for each given instance $\mu_3,\mu_4>0$

$$(g(u),u) \le \mu_3\|u\|_2^2 + \mu_4, \tag{4.13}$$

$$\begin{aligned}(g(u),w) &\ge \frac{d}{dt}\int_U G(u)dx + \varepsilon kG(u) - \varepsilon(\mu_1\|u\|_2^2+\mu_2) - |c||z(\theta_t\omega)|(\mu_3\|u\|_2^2+\mu_4). \\ &= \frac{d}{dt}\int_U G(u)dx + \varepsilon kG(u) - (\varepsilon\mu_1 + |c||z(\theta_t\omega)|\mu_3)\|u\|_2^2 - \varepsilon\mu_2 - \mu_4|c||z(\theta_t\omega)|.\end{aligned} \tag{4.14}$$

Where $\tilde{G}(u)=\int_U G(u)dx$. Collecting (4.5)-(4.14) and (4.4), showing that

$$\begin{aligned}(Q(\varphi,\omega,t),\varphi) \le& -\frac{d}{dt}\int_U G(u)dx - \varepsilon kG(u) + (\varepsilon\mu_1 + |c||z(\theta_t\omega)|\mu_3)\|u\|_2^2 + \varepsilon\mu_2 + \mu_4|c||z(\theta_t\omega) \\ &+\frac{c^2|z(\theta_t\omega)|^2}{2\sqrt{\lambda_0}}(\|u\|_2^2+\|w\|^2) + \frac{4\varepsilon^2|c|^2|z(\theta_t\omega)|^2}{\alpha\sqrt{\lambda_0}}\|u\|_2^2 + \frac{\alpha}{4}\|w\|^2 + \frac{2}{\alpha}\|f\|^2.\end{aligned} \tag{4.15}$$

Substituting all into (4.2) results in

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\varphi\|_E^2 + 2G(u)) + \frac{\varepsilon}{2}(\|u\|_2^2+\|w\|^2) + \frac{\delta}{4}\|\eta\|_{\mu,2}^2 + \frac{\alpha}{2}\|w\|^2 + \varepsilon kG(u) \\ &\le \frac{c^2|z(\theta_t\omega)|^2}{2\sqrt{\lambda_0}}(\|u\|_2^2+\|w\|^2) + (\frac{4\varepsilon^2|c|^2|z(\theta_t\omega)|^2}{\alpha\sqrt{\lambda_0}} + |c||z(\theta_t\omega)|\mu_3 + \varepsilon\mu_1)\|u\|_2^2 \\ &\quad+\frac{|c||z(\theta_t\omega)|}{2}\|u\|_2^2 + \frac{2}{\alpha}\|f\|^2 + \varepsilon\mu_2 + \mu_4|c|\ |z(\theta_t\omega)|.\end{aligned} \tag{4.16}$$

By defining $\sigma=\min\{\varepsilon,\varepsilon k,\frac{\delta}{2}\}$ and $\|\varphi\|^2=(\|u\|_2^2+\|w\|^2)$, the following equivalent system arises

$$\frac{1}{2}\frac{d}{dt}(\|\varphi\|_E^2 + 2G(u)) + \rho(t,\theta_t\omega)(\|\varphi\|_E^2 + 2G(u)) \le \frac{2}{\alpha}\|f\|^2 + \varepsilon\mu_2 + \mu_4|c||z(\theta_t\omega)|. \tag{4.17}$$

Where

$$\rho(t,\theta_t\omega) = \sigma - \mu_3|c||z(\theta_t\omega)| - (\frac{c^2|z(\theta_t\omega)|^2}{2\sqrt{\lambda_0}} + \frac{4\varepsilon^2|c|^2|z(\theta_t\omega)|^2}{\alpha\sqrt{\lambda_0}} + \varepsilon\mu_1 + \frac{|c||z(\theta_t\omega)|}{2}). \tag{4.18}$$

Using Gronwall's inequality on equation (4.17) over [0,t], it has been that

$$\begin{aligned}\|\varphi(t,\omega,\varphi_0)\|_E^2 + 2G(u) \leq\ & e^{-2\int_0^t \rho(s,\theta_s\omega)ds}[\|\varphi_0\|_E^2 + 2G(u_0)] \\ & +(\frac{2}{\alpha}\|f\|^2 + \varepsilon\mu_2)\int_0^t e^{-\int_s^t \rho(\tau,\theta_\tau\omega)d\tau}ds \\ & +\mu_4|c|\int_0^t e^{-\int_s^t \rho(\tau,\theta_\tau\omega)d\tau}|z(\theta_t\omega)|ds. \end{aligned} \tag{4.19}$$

Substitiuting ω by $\theta_{-t}\omega$, from (4.19) as a result

$$\begin{aligned}\|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2 + 2G(u) \leq\ & e^{-2\int_0^t \rho(s-t,\theta_{s-t}\omega)ds}[\|\varphi_0(\theta_{-t}\omega)\|_E^2 + 2G(u_0)] \\ & +(\frac{2}{\alpha}\|f\|^2 + \varepsilon\mu_2 +)\int_0^t e^{-\int_s^t \rho(\tau-t,\theta_{\tau-t}\omega)d\tau}ds \\ & +\mu_4|c|\int_0^t e^{-\int_s^t \rho(\tau-t,\theta_{\tau-t}\omega)d\tau}|z(\theta_{s-t}\omega)|ds. \\ \leq\ & e^{\int_{-t}^0 \rho(s,\theta_s\omega)ds}\|\varphi_0\|_E^2 \\ & +(\frac{2}{\alpha}\|f\|^2 + \varepsilon\mu_2)\int_{-t}^0 e^{-\int_s^0 \rho(\tau,\theta_\tau\omega)d\tau}ds \\ & +\mu_4|c|\int_{-t}^0 e^{-\int_s^0 \rho(\tau,\theta_\tau\omega)d\tau}|z(\theta_s\omega)|ds. \end{aligned} \tag{4.20}$$

As stated in (4.18), it is understood that

$$|c|(\mu_3 + \frac{1}{2})\frac{1}{\alpha} + c^2(\frac{1}{2\sqrt{\lambda_0}}\frac{1}{\sqrt{2}\alpha} + \frac{4\varepsilon^2}{\alpha\sqrt{\lambda_0}}\frac{1}{\sqrt{2}\alpha}) < \sigma. \tag{4.21}$$

Note that (4.12) and (4.13) lead to the conclusion

$$kG(u) \leq (g(u),u) + \mu_1\|u\|_2^2 + \mu_2 \leq (\mu_1+\mu_3)\|u\|_2^2 + \mu_2 + \mu_4. \tag{4.22}$$

It follows from Lemma 4.1, $\varphi_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, and , and the tempered property of $B(\omega)$

$$\lim_{t\to+\infty} e^{2\int_{-t}^0 -\rho(\tau,\theta_\tau\omega)d\tau}[\|\varphi_0(\theta_{-t}\omega)\|_\tau^2 + 2G(u_0)] = 0. \tag{4.23}$$

The following integral converges, as $|z(\theta_s\omega)|$ is tempered

$$\rho^2(\omega) = (\frac{2}{\alpha}\|f\|^2 + \varepsilon\mu_2 + c)\int_{-\infty}^0 e^{-\int_s^0 \rho(\tau,\theta_\tau\omega)d\tau}(1 + |z(\theta_s\omega)|)ds. \tag{4.24}$$

Lemma 3.1 and the fact that $g \in L^2(U)$, yield

$$\|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2 \leq \rho^2(\omega).$$

From (4.22)-(4.23) and Lemma 4.1, there exists a closed measurable absorbing ball $B_0(\omega) = \{\varphi \in E : \|\varphi_0(\theta_{-t}\omega)\|_E \leq \rho^2(\omega)\}$ with a positive time $T = T(0,B,\omega) > 0$ such that $\varphi(t,\theta_{-t}\omega,\varphi_0) = \varphi_0 \in B_0(\omega) \in \mathcal{D}(E)$ holds p-a.s. for $\omega \in \Omega$

$$\|\varphi(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_E^2 \leq \rho^2(\omega),$$

is complete the proof. □

5 Decomposition of Solutions

To obtain regularity estimates later, the nonlinear term in equation (3.3) was decomposed as in[28, 40, 41]. At first, the following decomposition given on nonlinearity $g(u) = g_1(u) + g_2(u)$ where $g_1, g_2 \in \mathbb{C}^1$ functions. These functions satisfy the following conditions for some proper constant:

$$\begin{cases} |g_1(s)| \le C(|s| + |s|^5), \ \forall \ s \in \ \mathbb{R}, \\ sg_1(s) \ge 0 \ , \\ \exists \ \rho_2, \ \vartheta_1 \ge 0 \ such \ that \ \forall \ \vartheta \in (0, \vartheta_1], \\ \exists \ c_\vartheta \in \mathbb{R}, \ \rho_2 G_1(s) + \vartheta s^2 - c_\vartheta \le sg_1(s), \ \forall \ s \in \mathbb{R}, \end{cases} \tag{5.1}$$

and

$$\begin{cases} |g_2'(s)| \le C(1 + |s|^p), \ \forall \ s \in \ \mathbb{R}, 0 < p < 5, \\ 3G_2(s) - C \le sg_2(s), \\ -\dfrac{\lambda}{8}s^2 - C \le G_2(s), \ \forall \ s \in \mathbb{R}, \end{cases} \tag{5.2}$$

where
$G_i(s) = \int_0^s g_i(r)dr, i = 1, 2.$
The solution $\varphi = (u, w)^T$ of the system (3.15) was decomposed into two parts,

$$\varphi = \varphi_L + \varphi_N$$

where $\varphi_L = (u_L, w_L)$, $\varphi_N = (u_N, w_N)$ respectively solves the following equations

$$\begin{cases} \varphi_L' + H(\varphi_L) + Q_1(\varphi_L) = 0, \\ \varphi_L(0, \omega) = (u_0, u_1 + \varepsilon u_0 - cu_0 z(\theta_t\omega))^T, \ t \ge 0, \end{cases} \tag{5.3}$$

and

$$\begin{cases} \varphi_N' + H(\varphi_N) + Q_2(\varphi, \varphi_L) = \tilde{Q_2}(\omega), \\ \varphi_N(0, \omega) = (0, \varepsilon z(\theta_t\omega), 0)^T, \ t \ge 0, \end{cases} \tag{5.4}$$

where

$$Q_1(\varphi_L) = \begin{pmatrix} 0 \\ g_1(u_L) \\ 0 \end{pmatrix}, \ Q_2(\varphi, \varphi_L) = \begin{pmatrix} 0 \\ g(u) - g_1(u_L) \\ 0 \end{pmatrix},$$
$$\tilde{Q_2}(\omega) = \begin{pmatrix} cu_N z(\theta_t\omega) \\ -cz(\theta_t\omega)(v_N - 2\varepsilon u_N + cu_N z(\theta_t\omega)) - g(u) + f(x) \\ cu_N z(\theta_t\omega) \end{pmatrix}. \tag{5.5}$$

To prove the existence of a compact random attractor for the RDS Φ, it is shown that the solutions of systems (5.3) and (5.4) are similar to the solution of system (4.2), with one decaying exponentially and the other being bounded in a higher regular space. In order to obtain the regularity estimate, some a priori estimate for the solutions of system (5.3) on $\Omega \times [0, \infty]$ will be proven.

Lemma 5.1. *Consider a bounded non-random subset B of E, for any $\varphi_L(0, \omega) = (u_0, u_1 + \varepsilon u_0 - cu_0 z(\theta_t\omega))^T \in B$, there holds*

$$\|\varphi_L(0, \omega; \varphi_L(0, \omega))\|_E^2 \le r_3^2(\omega), \tag{5.6}$$

where $\varphi_L = (u_L, v_L)^T$ satisfies (5.3).

Proof. By taking the inner product $(\cdot, \cdot)_E$ of (5.3) in $L^2(U)$ with $\varphi_L = (u_L, v_L)^T$, where $v_L = u_{Lt} + \varepsilon u_L$, and using initial values $(u_0, u_1 + \varepsilon u_0 - cu_0 z(\theta_t\omega))^T$, it follows that.

$$\frac{1}{2}\frac{d}{dt}\|\varphi_L\|_E^2 + (H(\varphi_L), \varphi_L)_E + (Q_1(\varphi_L), \varphi_L) = 0, \tag{5.7}$$

there holds after a simple computation

$$(H(\varphi_L),\varphi_L)_E \geq \frac{\varepsilon}{2}\left(\|u_L\|_2^2+\|v_L\|^2\right)+\frac{\alpha}{2}\|v_L\|^2, \tag{5.8}$$

given that ε satisfies (4.4), the third term of (5.7) can now be estimated as.

$$\begin{aligned}(Q(\varphi_L),\varphi_L) = &\begin{pmatrix} 0 \\ g_1(u_L)\end{pmatrix}\begin{pmatrix} u_L \\ v_L\end{pmatrix}\\ &= (g_1(u_L), u_{Lt}+\varepsilon u_L)\\ &= \tfrac{d}{dt}G_1(u_L)+\varepsilon\textstyle\int_U g_1(u_L)u_L dx.\end{aligned} \tag{5.9}$$

As a result of $(5.1)_2$ and $(5.1)_3$, it follows

$$\begin{aligned}&G_1(u_L)\geq 0\ ,\ g_1(u_L)u_L\geq 0,\\ &\tfrac{d}{dt}G_1(u_L)+\varepsilon\textstyle\int_U g_2(u_L)u_L dx \geq \tfrac{d}{dt}G_1(u_L)+k_0\varepsilon G_1(u_L)+\varepsilon\vartheta\|u_L\|^2-\varepsilon c_\vartheta.\end{aligned} \tag{5.10}$$

By combining (5.7)-(5.10) and (5.3), it follows

$$\frac{d}{dt}\left(\|\varphi_L\|_E^2+2\tilde{G}_1(u_L)\right)+2\sigma_L\left(\|\varphi_L\|_E^2+2\tilde{G}_1(u_L)\right)\leq\rho, \tag{5.11}$$

whereas $\rho=\varepsilon c_\vartheta$ and $\sigma_L=\min(\frac{\varepsilon}{2},\frac{\alpha}{2},\frac{\varepsilon}{4},k_0\varepsilon)$

$$\|\varphi_L\|_E^2+2\tilde{G}_1(u_L)\geq\|\varphi_L\|_E^2\geq 0, \tag{5.12}$$

hence

$$\begin{aligned}\varphi_{L(0,\omega)} &= (\varphi_0(\theta_{-t}\omega)+cuz(\theta_{-t}\omega))^\top\\ &\leq (r_2(\omega)+cuz(\theta_t\omega))=\rho_2(\omega)\in B_0(\omega).\end{aligned} \tag{5.13}$$

By combining (5.1)1, (5.11), and (5.13), and applying Gronwall's inequality to the result over $[0,t]$, it can be proven with the definition of $B_0(\omega)$ and Lemma 4.2.

$$\|\varphi_L(0,\omega,\varphi_{L(\tau,\omega)})\|_E\leq r_3^2(\omega). \tag{5.14}$$

The proof is completed. □

Lemma 5.2. *There exists a positive constant $\sigma_1\geq 0$, such that for any bounded non-random subsetB of E, it holds that for any $\varphi_L(0,\omega)=(u_0,u_1+\varepsilon u_0-cu_0z(\theta_t\omega))^T\in B$, we have*

$$\|\varphi_L(0,\omega;\varphi_L(0,\omega))\|_E^2\leq r_4^2(\omega)e^{2\sigma_1(\omega)t}, t\geq 0, \tag{5.15}$$

whereas $\varphi_L=(u_L,v_L)^T$ satisfies (5.3).

Proof. Like Lemma 5.1, consider equation (5.7). According to (5.1), $(g_1(u_L),(u_L))\geq 0,\ g_1(0)=0$ has a non-negative value of. By applying the Sobolev embedding theorem $H^1\subset L^6\subset L^4\subset L^2$ and using (5.6), a conclusion can be drawn

$$\begin{aligned}0\leq\tilde{G}_1(u_L)&\leq\int_U G_1(u^1)dx\\ &\leq C(\|u_L\|^2+\|u_L\|_{L^6}^6)\\ &\leq\rho_2(\omega)\|u_L\|_1^2,\\ \sigma_1\|u_L\|_1^2&\geq\tfrac{\sigma_1}{\rho_2(\omega)}\tilde{G}_1(u_L),\ \forall\ u_L\in\mathbb{R},\end{aligned} \tag{5.16}$$

As a result of (5.7) and (5.16), the following conclusion can be drawn

$$\frac{d}{dt}(\|\varphi_L\|_E^2+2\tilde{G}_1(u_L))+2\sigma_1\|\varphi_L\|_E^2+\frac{\sigma_1}{2\rho_2(\omega)}\tilde{G}_1(u_L)\leq\rho. \tag{5.17}$$

Since $\sigma_1(\omega) = \min\ [\sigma_1, \frac{\sigma_1}{2\rho_2(\omega)}]$.
By utilizing Gronwall's inequality on equation (5.17), the following result is obtained

$$\begin{aligned}\|\varphi_L(0,\omega,\varphi_L(0,\omega)\|_E^2 &\leq \left(\|\varphi_L(0,\omega)\|_E^2 + \tilde{G}_1(u_L(0))\right) e^{2\sigma_1(\omega)t} + \rho \int_0^t e^{-2\sigma_1(s,\omega)} ds \\ &\leq \left(\rho_1^2(\omega) + \tilde{G}_1(u_L(0))\right) e^{2\sigma_1(\omega)t} + \rho \int_0^t e^{-2\sigma_1(s,\omega)} ds, \\ &\leq r_4^2(\omega)\end{aligned} \tag{5.18}$$

By using $(5.1)_1$, the following estimate can be obtained

$$\tilde{G}_1(u_L) = \int_U G_1(u^1)dx \leq C(\|u_L\|^2 + \|u_L\|_{L^6}^6) \leq C_g\|u_L\|_{H^1}^6 \leq C_p{\rho_1}^6(\omega),\ \forall\ u_L \in \mathbb{R}. \tag{5.19}$$

By combining all equations (5.13) and (5.18)-(5.19), the final result of (5.15) can be obtained

$$r_4^2(\omega) \leq \left(\rho_1^2(\omega) + C_p{\rho_1}^6(\omega)\right) e^{2\sigma_1(\omega)t} + \rho \int_0^t e^{-2\sigma_1(s,\omega)} ds.$$

The proof is completed. □

Lemma 5.3. *Assume that $(h_1) - (h_3)$ hold, and (5.1)-(5.2) are satisfied, there exists a random radius $r_5(\omega)$, such that for P-a.e.$\omega \in \Omega$, it holds.*

$$\left\|A^{\frac{1+\nu}{2}} u_N\right\|^2 + \left\|A^{\frac{\nu}{2}} u_{Nt}\right\|^2 \leq r_5(\omega), \tag{5.20}$$

whereas

$$\nu = min\{\frac{1}{4}, \frac{4-p}{4}\},\ \forall\ 0 \leq p \leq 4. \tag{5.21}$$

Proof. According to (5.6), (4.1), and the definition of $\varphi_N = \varphi - \varphi_L$, there exists a random variable $r(\omega) > 0$ such that

$$\max\{\|\varphi(0,\omega,\varphi(0,\omega))\|_E, \|\varphi_N((0,\omega,\varphi_N(0,\omega)))\|_E\} \leq r(\omega). \tag{5.22}$$

By taking the inner product of (5.4) with $(A^\nu\varphi_N, A^\nu w_N)^T$ using the inner product $(\cdot,\cdot)_E$, it can be found that.

$$\left(\varphi_N', A^\nu\varphi_N\right) + (H(\varphi_N), A^\nu\varphi_N) = \left(\tilde{Q}_2(\varphi_N,\omega,t), A^\nu\varphi_N\right) \tag{5.23}$$

By using (5.21) and referring to Lemma 4.1, the following result can be obtained

$$(H(\varphi_N), A^\nu\varphi_N)_E \geq \frac{\varepsilon}{2}\left(\left\|A^{\frac{1+\nu}{2}} u_N\right\|_2^2 + \left\|A^{\frac{\nu}{2}} w_N\right\|^2\right) + \frac{\alpha}{2}\left\|A^{\frac{\nu}{2}} w_N\right\|^2, \tag{5.24}$$

Next, the right-hand side of equation (5.23) will be estimated, resulting in

$$\begin{aligned}\left(\tilde{Q}_2(\varphi_N,\omega,t), A^\nu\varphi_N\right) = &\\ &((cu_N z(\theta_t\omega), A^\nu u_N)) - (cw_N z(\theta_t\omega), A^\nu w_N) \\ &+ (2c\varepsilon u_N z(\theta_t\omega), A^\nu w_N) - (c^2 u_N z^2(\theta_t\omega), A^\nu w_N) \\ &- (g(u) - g_1(u_L), A^\nu w_N) + (f(x), A^\nu w_N).\end{aligned} \tag{5.25}$$

Now, the right term in equation (5.25) will be handled by utilizing (4.5) to (4.10) and (5.21), leading to

$$((cu_N z(\theta_t\omega), A^\nu u_N)) \leq |c||z(\theta_t\omega)|\|A^{\frac{1+\nu}{2}} u_N\|^2, \tag{5.26}$$

$$(cw_N z(\theta_t\omega), A^\nu w_N) \leq |c||z(\theta_t\omega)|\|A^{\frac{\nu}{2}} w_N\|^2, \tag{5.27}$$

$$(2c\varepsilon u_N z(\theta_t\omega), A^\nu w_N) \leq \frac{\varepsilon|c||z(\theta_t\omega)|}{2\sqrt{\lambda_0}}\left(\|A^{\frac{1+\nu}{2}}u_N\|^2 + \|A^{\frac{\nu}{2}}w_N\|^2\right), \tag{5.28}$$

$$(c^2u_N z^2(\theta_t\omega), A^\nu w_N) \leq \frac{|c|^2|z(\theta_t\omega)|^2}{2\sqrt{\lambda_0}}(\|A^{\frac{1+\nu}{2}}u_N\|^2 + \|A^{\frac{\nu}{2}}w_N\|^2), \tag{5.29}$$

$$(f(x), A^\nu w_N) \leq \frac{1}{\alpha}\|A^{\frac{\nu}{2}}f(x)\|^2 + \frac{\alpha}{4}\|A^{\frac{\nu}{2}}w_N\|^2. \tag{5.30}$$

For the nonlinear term, it is straightforward to demonstrate that

$$\begin{aligned}(g(u) - g_1(u_L), A^\nu w_N) &= (g(u) - g_1(u_N), A^\nu(u_N + \varepsilon u_N - cu_N z(\theta_t\omega)))\\ &\leq \tfrac{d}{dt}\textstyle\int_U (g(u) - g_1(u_L))A^\nu u_N dx + \int_U (g(u) - g_1(u_L))A^\nu u_N dx\\ &\quad - \textstyle\int_U (g'(u)u_t - g_1'(u_L)u_{Lt})A^\nu u_N dx - C\int_U (g(u) - g_1(u_L))A^\nu u_N z(\theta_t\omega)dx,\end{aligned}$$

Next, by using (1.2), (5.1) to (5.2), the Cauchy-Schwartz inequality, and the Young inequality, the following conclusion can be reached

$$\int_U (g'(u)u_t - g_1'(u_L)u_{Lt})A^\nu u_N dx = \int_U ((g_1'(u) - g_1'(u_L))u_t + g_1'(u_L)u_{Nt} + g_2'(u)u_t)A^\nu u_N dx, \tag{5.31}$$

As a result, the following inequalities are obtained

$$\begin{aligned}\int_U (g_1'(u) - g_1'(u_L))u_t A^\nu u_N dx &\leq C\int_U g_1''(u + \theta(u - u_L))|u - u_L||u_t||A^\nu u_N|dx\\ &\leq C\textstyle\int_U \left(1 + |u|^3 + |u_L|^3\right)|u_N||A^\nu u_N||u_t|dx\\ &\leq C\left(1 + \|u\|_{L^{10}}^3 + \|u_L\|_{L^{10}}^3\right)\|u_N\|_{L^{\frac{10}{1-4\nu}}}\|A^\nu u_N\|_{L^{\frac{10}{1+4\nu}}}\|u_t\|_{L^{10}}\\ &\leq k_1(\omega)\|A^{\frac{1+\nu}{2}}u_N\|\\ &\leq 4\varepsilon k_1^2(\omega) + \tfrac{\varepsilon}{16}\|A^{\frac{1+\nu}{2}}u_N\|^2,\end{aligned} \tag{5.32}$$

and note that $\nu \leq \frac{4-p}{4}$

$$\begin{aligned}\int_U g_2'(u)u_t A^\nu u_N dx &\leq C\int_U (1 + |u|^p)|u_t||A^\nu u_N|dx\\ &\leq C(1 + \|u\|^p_{L^{\frac{10}{4-4\nu}}})\|A^\nu u_N\|_{L^{\frac{10}{1+4\nu}}}\|u_t\|_{L^2}\\ &\leq C(1 + \|\nabla u\|_2^p)\|A^\nu u_N\|_{L^{\frac{10}{1+4\nu}}}\|u_t\|_{L^6}\\ &\leq 4\varepsilon k_2^2(\omega) + \tfrac{\varepsilon}{16}\|A^{\frac{1+\nu}{2}}u_N\|^2,\end{aligned} \tag{5.33}$$

$$\begin{aligned}\int_U g_1'(u_L)u_{Nt}A^\nu u_N dx &\leq C(1 + \|u_L\|_{L^{10}}^4)\|A^{\frac{1+\nu}{2}}u_N\|_{L^{\frac{10}{1+4\nu}}}\|A^\nu u_{Nt}\|_{L^{\frac{10}{1+4\nu}}}\\ &\leq C(1 + \|u_L\|_{L^{10}}^4)\|A^{\frac{1+\nu}{2}}u_N\|_{L^{\frac{10}{1+4\nu}}}\|A^\nu u_{Nt}\|_{L^{\frac{10}{1+4\nu}}}\\ &\leq 4\varepsilon k_3(\omega)(\|A^{\frac{\nu}{2}}u_N\|^2 + |\varepsilon|^2) + \tfrac{\varepsilon}{16}\|A^{\frac{1+\nu}{2}}u_N\|^2_{L^{\frac{10}{1+4\nu}}}\end{aligned} \tag{5.34}$$

and

$$\begin{aligned}&\int_U (g(u) - g_1(u_L))|A^\nu u_N||z(\theta_t\omega)|dx\\ &\leq C\textstyle\int_U g'(u + \theta(u - u_l))|u - uL||A^\nu u_N||z(\theta_t\omega)|dx\\ &\leq C\textstyle\int_U \left(1 + |u|^4 + |u_L|^4\right)|u_N||A^\nu u_N||z(\theta_t\omega)|dx\\ &\leq C\left(1 + \|u\|_{L^{10}}^4 + \|u_L\|_{L^{10}}^4\right)\|u_N\|_{L^{\frac{10}{1-4\nu}}}\|A^\nu u_N\|_{L^{\frac{10}{1+4\nu}}}|z(\theta_t\omega)|\\ &\leq 4\varepsilon\left(k_4^2(\omega) + |z(\theta_t\omega)|^2\right) + \tfrac{\varepsilon}{16}\left\|A^{\frac{1+\nu}{2}}u_N\right\|^2.\end{aligned} \tag{5.35}$$

By combining equations (5.24) to (5.35) into (5.23), it can be demonstrated that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|A^{\frac{\nu}{2}}\varphi_2\|_E^2 + 2(g(u) - g_1(u_L))\right) + \frac{\varepsilon}{4}\left\|A^{\frac{\nu}{2}}\varphi_2\right\|_E^2 + \frac{k\varepsilon}{2}(g(u) - g_1(u_L))\\ &\leq \mu_2|c||z(\theta_t\omega)|\|A^{\frac{\nu}{2}}\varphi_2\|_E^2 + C(\omega)[1 + k_1^2(\omega) + k_2^2(\omega)\\ &+ k_3^2(\omega) + k_4^2(\omega) + |z(\theta_t\omega)|^2 + |z(\theta_t\omega)|^4 + \|A^{\frac{\nu}{2}}f(x)\|^2].\end{aligned} \tag{5.36}$$

By applying Gronwall's inequality to equation(5.36), the following result is obtained.

$$\begin{aligned}&\left\|A^{\frac{\nu}{2}}\varphi_2(t,\omega,\varphi(0,\omega))\right\|_E^2\\ &\le \left(\|A^{\frac{\nu}{2}}\varphi_2(0,\omega,\varphi(0,\omega))\|_E^2 + 2(g(u(0,\omega,\varphi(0,\omega))) - g_1(u_L\left(0,\omega,\varphi(0,\omega)\right)\right)\\ &\le \left(\|A^{\frac{\nu}{2}}\varphi_2\|_E^2 + (g(u) - g_1(u_L))\right) e^{2\int_0^t(\sigma-\mu_2|c||z(\theta_s\omega)|))(s,\omega)ds}\\ &\quad + \int_{-t}^0 \rho_1(\omega)e^{2\int_0^s(\sigma-\mu_2|c||z(\theta_\varsigma\omega)|)(\varsigma,\omega)d\varsigma}ds.\end{aligned} \tag{5.37}$$

put

$$\begin{aligned}\rho_1(\omega) = C(\omega)[1 + k_1^2(\omega) + k_2^2(\omega) + k_3^2(\omega) + k_4^2(\omega)\\ + |z(\theta_t\omega)|^2 + |z(\theta_t\omega)|^4 + \|A^{\frac{\nu}{2}}f(x)\|^2]\end{aligned} \tag{5.38}$$

similar to above equation

$$\begin{aligned}\int_U (g(u) - g_1(u_L))\, A^\nu u_N dx &\le C\int_U (g'(u + \theta(u-u_L))|u-u_L||A^\nu u_N|dx\\ &\le C\int_U \left(1 + |u|^4 + |u_L|^4\right)|u_N||A^\nu u_N|dx\\ &\le C(1 + \|u\|_{L^{10}}^4 + \|u_L\|_{L^{10}}^4)\|u_N\|_{L^{\frac{10}{1-4\nu}}}\|A^\nu u_N\|_{L^{\frac{10}{1+4\nu}}}\\ &\le k_5(\omega)\,\|A^{\frac{1+\nu}{2}}u_N\|\|A^\nu u_N\|\\ &\le \varepsilon k_5^2(\omega)\,\|A^{\frac{\nu}{2}}u_N\|^2 + \frac{\varepsilon}{4}\|A^{\frac{1+\nu}{2}}u_N\|^2,\end{aligned} \tag{5.39}$$

by (5.38) and (5.39), to get

$$\|A^\nu\varphi_2(t,\omega,\varphi(0,\omega))\|_E^2 \le r_5(\omega),$$

this complete the proof. □

6 Random Attractors

In this section, the existence of a $\mathcal{D}$-random attractor for the random dynamical system Φ associated with system (3.15) on $\mathbb{R}^5$ is established. This is done by using Lemma 4.1, which shows that, Φ has a closed random absorbing set in $\mathcal{D}$. This, combined with the $\mathcal{D}$-pullback asymptotic compactness, implies the existence of a unique $\mathcal{D}$-random attractor. The $\mathcal{D}$-pullback asymptotic compactness of Φ will be further demonstrated through the decomposition of solutions, as discussed in[40, 41, 47, 48].

Lemma 6.1. *assume that $(h_1) - (h_3)$ holds, it can be concluded that the random dynamical system (RDS) Φ associated with equation (3.5) has a uniformly attracting set $\Lambda(0,\omega) \subset E$, and a random attractor $\mathcal{A}(0,\omega) \subseteq \Lambda(0,\omega) \cap B_0(\omega)$, for any time $t \ge 0$ and any value of $\omega \in \Omega$.*

Proof. For all $t \ge 0, \omega \in \Omega$, in accordance with Lemma 5.3, define $B_\nu(0,\omega)$ as the closed ball in $H_{2+2\nu} \times H_{2\nu}$ with radius $r_5(\omega)$

$$\Lambda(0,\omega) = B_\nu(0,\omega), \tag{6.1}$$

next, $\Lambda(0,\omega) \in \mathfrak{D}(E)$. Since $H_{2+2\nu} \times H_{2\nu} \hookrightarrow H_0^2(U) \times L^2(U)$, it is now necessary to demonstrate the attractive property of $\Lambda(0,\omega)$: for every $B(0,\omega) \in \mathfrak{D}(E)$,

$$\lim_{t\to\infty} d_H(\Phi(t,\theta_{-t}\omega, B(0,\theta_{-t}\omega)), \Lambda(0,\omega)) = 0. \tag{6.2}$$

As per Lemma 5.2, this implies that

$$\varphi_N(0,\omega,\varphi(0,\omega)) = \varphi(0,\omega,\varphi(0,\omega)) - \varphi_L(0,\omega,\varphi(0,\omega)) \in \Lambda(0,\omega). \tag{6.3}$$

Therefore, Lemma 5.2 yields

$$\inf_{\psi \in \Lambda(0,\omega)} \|\varphi(0,\omega,\varphi(0,\omega)) - \psi\|_E^2 \leq \|\varphi_L(0,\omega,\varphi_L(0,\omega))\|_E^2 \leq r_4^2(\omega)e^{2\sigma_1(\omega)t},\ t \geq 0. \tag{6.4}$$

However, for all $t > 0$

$$dist(\Phi(t,\theta_{-t}\omega, B(0,\theta_{-t}\omega)), \Lambda(0,\omega)) \leq r_4^2(\omega)e^{-2\sigma_1(\omega)t}. \tag{6.5}$$

Finally, it is easy to see from the relationship between Φ and Ψ that for any non-random, bounded set $B \subset E$, it holds true with probability P-a.s.

$$dist(\Psi(t,\theta_{-t}\omega, B(0,\theta_{-t}\omega)), \Lambda(0,\omega)) \to 0, t \to +\infty. \tag{6.6}$$

As a result, the random dynamical system Φ connected to (3.5) has a random attractor $\mathcal{A}(0,\omega) \subseteq \Lambda(0,\omega) \cap B(\omega)$, $\mathcal{A} = \{\mathcal{A}(0,\omega) : t \geq 0, \omega \in \Omega\} \in \Omega$ in $\mathbb{R}^5$.
Then the proof is completed. □

Theorem 6.2. *Assuming $(h_1) - (h_3)$ hold, the continuous cocycle Φ associated with problem (3.8) or random dynamical system Φ has a unique $\mathcal{D}$-pullback attractor $A = \{A(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $\mathbb{R}^5$.*

Proof. According to Lemma 4.2, the continuous cocycle Φ has a closed random absorbing set $\{A(\omega)\}_{\omega\in\Omega}$ in $\mathcal{D}$. Additionally, as per (3.17) and Lemma 6.1, the continuous cocycle Φ is $\mathcal{D}$-pullback asymptotically compact in $\mathbb{R}^5$. Thus, the existence of a unique $\mathcal{D}$- random attractor for Φ is a direct result of Lemma 2.1. □

7 Conclusion

To summarize, by Lemma 4.1, Φ has a closed random absorbing set in $\mathcal{D}$. This, combined with the $\mathcal{D}$-pullback asymptotic compactness, implies the existence of a unique $\mathcal{D}$-random attractor. We established the $\mathcal{D}$-pullback asymptotic compactness of Φ through the decomposition of solutions and proved the existence of a $\mathcal{D}$-random attractor for the random dynamical system Φ associated with system (3.15) in $\mathbb{R}^5$. □

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Competing Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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