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Generalized Friedrich Numbers

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Abstract

In this paper, we define and investigate the generalized Friedrich sequences and we deal with, in detail, two special cases, namely, Friedrich and Friedrich-Lucas sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there are close relations between Friedrich, Friedrich-Lucas and third order Jacobsthal, modified third-order Jacobsthal, third order Jacobsthal-Lucas numbers.

Keywords: Friedrich numbers; friedrich-Lucas numbers; Jacobsthal numbers; third order Jacobsthal numbers; modified third-order Jacobsthal numbers; third order Jacobsthal-Lucas numbers.

2020 Mathematics Subject Classification: 11B37, 11B39, 11B83.

1 Introduction

Third-order Jacobsthal sequence $\{J_n\}_{n\geq 0}$ (OEIS: A077947, [1]), modified third-order Jacobsthal sequence $\{K_n\}_{n\geq 0}$ (OEIS: A186575, [1]) and third-order Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ (OEIS: A226308, [1]) are defined,

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respectively, by the third-order recurrence relations

$$J_{n+3} = J_{n+2} + J_{n+1} + 2J_n, \quad J_0 = 0, J_1 = 1, J_2 = 1, \tag{1.1}$$

$$K_{n+3} = K_{n+2} + K_{n+1} + 2K_n, \quad K_0 = 3, K_1 = 1, K_2 = 3. \tag{1.2}$$

$$j_{n+3} = j_{n+2} + j_{n+1} + 2j_n, \quad j_0 = 2, j_1 = 1, j_2 = 5, \tag{1.3}$$

The sequences $\{J_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ are defined in [2] and $\{K_n\}_{n\geq 0}$ is given in [3]. For more details on the generalized third-order Jacobsthal numbers and its special cases, see [4].

The sequences $\{J_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} J_{-n} &= -\frac{1}{2}J_{-(n-1)} - \frac{1}{2}J_{-(n-2)} + \frac{1}{2}J_{-(n-3)}, \\ K_{-n} &= -\frac{1}{2}K_{-(n-1)} - \frac{1}{2}K_{-(n-2)} + \frac{1}{2}K_{-(n-3)}, \\ j_{-n} &= -\frac{1}{2}j_{-(n-1)} - \frac{1}{2}j_{-(n-2)} + \frac{1}{2}j_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.1)-(1.3) hold for all integer n.

Now, we define two sequences related to third-order Jacobsthal, modified third-order Jacobsthal and third-order Jacobsthal-Lucas numbers. Friedrich and Friedrich-Lucas numbers are defined as

$$F_n = F_{n-1} + F_{n-2} + 2F_{n-3} + 1, \quad \text{with} \quad F_0 = 0, F_1 = 1, F_2 = 2, \quad n \geq 3, \tag{1.4}$$

and

$$C_n = C_{n-1} + C_{n-2} + 2C_{n-3} - 3, \quad \text{with} \quad C_0 = 4, C_1 = 2, C_2 = 4, \quad n \geq 3, \tag{1.5}$$

respectively. The first few values of Friedrich and Friedrich-Lucas numbers are

$$0, 1, 2, 4, 9, 18, 36, 73, 146, 292, 585, 1170, 2340, 4681, ...$$

and

$$4, 2, 4, 11, 16, 32, 67, 128, 256, 515, 1024, 2048, 4099, 8192, ...$$

respectively. The sequences $\{F_n\}$ and $\{C_n\}$ satisfy the following fourth order linear recurrences:

$$F_n = 2F_{n-1} + F_{n-3} - 2F_{n-4}, \quad F_0 = 0, F_1 = 1, F_2 = 2, F_3 = 4, \quad n \geq 4, \tag{1.6}$$

$$C_n = 2C_{n-1} + C_{n-3} - 2C_{n-4}, \quad C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 11, \quad n \geq 4. \tag{1.7}$$

There are close relations between Friedrich, Friedrich-Lucas and third-order Jacobsthal, modified third-order Jacobsthal, third-order Jacobsthal-Lucas numbers. For example, they satisfy the following interrelations:

$$\begin{aligned} 3F_n &= J_{n+2} + 2J_n - 1, \\ 2C_n &= -J_{n+2} + 7J_{n+1} - 3J_n + 2, \\ 147F_n &= 17K_{n+2} + 10K_{n+1} - 4K_n - 49, \\ C_n &= K_n + 1, \\ 18F_n &= j_{n+2} + 3j_{n+1} - j_n - 6, \\ 24C_n &= 11j_{n+2} - 21j_{n+1} + 19j_n + 24, \end{aligned}$$

and

$$\begin{aligned} J_{n+1} &= F_{n+1} - F_n, \\ 147J_n &= 19C_{n+2} - 9C_{n+1} - 16C_n + 6, \\ 4K_n &= F_{n+2} + 13F_{n+1} - 23F_n - 3, \\ 3K_n &= C_{n+3} - C_{n+2} - C_{n+1} + C_n, \\ j_n &= -F_{n+3} + 3F_{n+2} - 2F_n, \\ 49j_n &= -5C_{n+2} + 23C_{n+1} + 30C_n - 48. \end{aligned}$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., Friedrich, Friedrich-Lucas numbers). First, we recall some properties of the generalized Tetranacci numbers.

The generalized (r,s,t,u) sequence (or generalized Tetranacci sequence or generalized 4-step Fibonacci sequence) $\{W_n(W_0,W_1,W_2,W_3;r,s,t,u)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \qquad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, \quad n \geq 4 \tag{1.8}$$

where W_0,W_1,W_2,W_3 are arbitrary complex (or real) numbers and r,s,t,u are real numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [5,6,7,8,9,10,11,12,13]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1,2,3,...$ when $u \neq 0$. Therefore, recurrence (1.8) holds for all integers n.

As $\{W_n\}$ is a fourth-order recurrence sequence (difference equation), its characteristic equation is

$$z^4 - rz^3 - sz^2 - tz - u = 0 \tag{1.9}$$

whose roots are $\alpha,\beta,\gamma,\delta$. Note that we have the following identities

$$\begin{aligned}
\alpha+\beta+\gamma+\delta &= r,\\
\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= -s,\\
\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta &= t,\\
\alpha\beta\gamma\delta &= -u.
\end{aligned}$$

Using these roots and the recurrence relation, Binet's formula can be given as follows (which can be found in the literature, for completeness, we include the proof):

Theorem 1.1. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$) For all integers n, Binet's formula of generalized Tetranacci numbers is*

$$W_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{1.10}$$

where

$$\begin{aligned}
p_1 &= W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0,\\
p_2 &= W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0,\\
p_3 &= W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0,\\
p_4 &= W_3 - (\alpha+\beta+\gamma)W_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)W_1 - \alpha\beta\gamma W_0.
\end{aligned}$$

Proof. If the roots $\alpha,\beta,\gamma,\delta$ of (1.10) are distinct, then (the sequences $(\alpha^n)_{n\geq 0}$, $(\beta^n)_{n\geq 0}$, $(\gamma^n)_{n\geq 0}$ and $(\delta^n)_{n\geq 0}$ are solutions of (1.8) and) the general formula of W_n is in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n$$

where the coefficients A_1, A_2, A_3 and A_4 are determined by the system of linear equations

$$\begin{aligned}
W_0 &= A_1 + A_2 + A_3 + A_4\\
W_1 &= A_1\alpha + A_2\beta + A_3\gamma + A_4\delta\\
W_2 &= A_1\alpha^2 + A_2\beta^2 + A_3\gamma^2 + A_4\delta^2\\
W_3 &= A_1\alpha^3 + A_2\beta^3 + A_3\gamma^3 + A_4\delta^3
\end{aligned}$$

Solving these four simultaneous equations for W_0, W_1, W_2 and W_3, we obtain the required result. □

Usually, it is customary to choose $\alpha, \beta, \gamma, \delta$ so that the Equ. (1.9) has at least one real (say α) solutions. Note that the Binet form of a sequence satisfying (1.9) for non-negative integers is valid for all integers n (see [14]).

Next, we consider two special cases of the generalized (r,s,t,u) sequence $\{W_n\}$ which we call them (r,s,t,u)-Fibonacci and (r,s,t,u)-Lucas sequences. (r,s,t,u)-Fibonacci sequence $\{G_n\}_{n\geq 0}$ and (r,s,t,u)-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned} G_{n+4} &= rG_{n+3}+sG_{n+2}+tG_{n+1}+uG_n, \quad (1.11)\\ G_0 &= 0, G_1=1, G_2=r, G_3=r^2+s,\\ H_{n+4} &= rH_{n+3}+sH_{n+2}+tH_{n+1}+uH_n, \quad (1.12)\\ H_0 &= 4, H_1=r, H_2=2s+r^2, H_3=r^3+3sr+3t. \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{t}{u}G_{-(n-1)}-\frac{s}{u}G_{-(n-2)}-\frac{r}{u}G_{-(n-3)}+\frac{1}{u}G_{-(n-4)},\\ H_{-n} &= -\frac{t}{u}H_{-(n-1)}-\frac{s}{u}H_{-(n-2)}-\frac{r}{u}H_{-(n-3)}+\frac{1}{u}H_{-(n-4)}, \end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (1.11) and (1.12) hold for all integers n.

For all integers n, (r,s,t,u)-Fibonacci and (r,s,t,u)-Lucas numbers (using initial conditions in (1.11) or (1.12)) can be expressed using Binet's formulas as in the following corollary (by setting $W_n=G_n$ and $W_n=H_n$ in Theorem 1.1, respectively).

Corollary 1.2. *(Four Distinct Roots Case: $\alpha\neq\beta\neq\gamma\neq\delta$) Binet's formula of (r,s,t,u)-Fibonacci and (r,s,t,u)-Lucas numbers are*

$$G_n=\frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

and

$$H_n=\alpha^n+\beta^n+\gamma^n+\delta^n,$$

respectively.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty}W_nz^n$ of the sequence W_n.

Lemma 1.3. *[9, Lemma 1] Suppose that $f_{W_n}(z)=\sum_{n=0}^{\infty}W_nz^n$ is the ordinary generating function of the generalized (r,s,t,u) sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty}W_nz^n$ is given by*

$$\sum_{n=0}^{\infty}W_nz^n=\frac{W_0+(W_1-rW_0)z+(W_2-rW_1-sW_0)z^2+(W_3-rW_2-sW_1-tW_0)z^3}{1-rz-sz^2-tz^3-uz^4}. \quad (1.13)$$

The following theorem presents Simson's formula of generalized (r,s,t,u) sequence (generalized Tetranacci sequence) $\{W_n\}$.

Theorem 1.4 (Simson's Formula of Generalized (r,s,t,u) Numbers)**.** *[15] For all integers n, we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n\\ W_{n+2} & W_{n+1} & W_n & W_{n-1}\\ W_{n+1} & W_n & W_{n-1} & W_{n-2}\\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0\\ W_2 & W_1 & W_0 & W_{-1}\\ W_1 & W_0 & W_{-1} & W_{-2}\\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \quad (1.14)$$

The following theorem shows that the generalized Tetranacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 1.5. *[16, Theorem 1.] For $n \in \mathbb{Z}$, for the generalized Tetranacci sequence (or generalized (r, s, t, u)-sequence or 4-step Fibonacci sequence) we have the following:*

$$\begin{aligned} W_{-n} &= \frac{1}{6}(-u)^{-n}(-6W_{3n} + 6H_nW_{2n} - 3H_n^2W_n + 3H_{2n}W_n + W_0H_n^3 + 2W_0H_{3n} - 3W_0H_nH_{2n}) \\ &= (-1)^{-n-1}u^{-n}(W_{3n} - H_nW_{2n} + \frac{1}{2}(H_n^2 - H_{2n})W_n - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_0). \end{aligned}$$

Using Theorem 1.5, we have the following corollary.

Corollary 1.6. *[16, Corollary 4] For $n \in \mathbb{Z}$, we have*

(a) $2(-u)^{n+4}G_{-n} = -(3ru^2 + t^3 - 3stu)^2G_n^3 - (2su - t^2)^2G_{n+3}^2G_n - (-rt^2 - tu + 2rsu)^2G_{n+2}^2G_n - (-st^2 + 2s^2u + 4u^2 + rtu)^2G_{n+1}^2G_n + 2(3ru^2 + t^3 - 3stu)((-2su + t^2)G_{n+3} + (-rt^2 - tu + 2rsu)G_{n+2} + (-st^2 + 2s^2u + 4u^2 + rtu)G_{n+1})G_n^2 + 2(2su - t^2)(-rt^2 - tu + 2rsu)G_{n+3}G_{n+2}G_n + 2(2su - t^2)(-st^2 + 2s^2u + 4u^2 + rtu)G_{n+3}G_{n+1}G_n - 2(-st^2 + 2s^2u + 4u^2 + rtu)(-rt^2 - tu + 2rsu)G_{n+2}G_{n+1}G_n - 2G_{3n}u^4 + u^2(-2su + t^2)G_{2n+3}G_n + u^2(-rt^2 - tu + 2rsu)G_{2n+2}G_n + u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n+1}G_n - 2u^2(2su - t^2)G_{2n}G_{n+3} + 2u^2(-rt^2 - tu + 2rsu)G_{2n}G_{n+2} + 2u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n}G_{n+1} - 3u^2(3ru^2 + t^3 - 3stu)G_{2n}G_n.$

(b) $H_{-n} = \frac{1}{6}(-u)^{-n}\left(H_n^3 + 2H_{3n} - 3H_{2n}H_n\right).$

Note that G_{-n} and H_{-n} can be given as follows by using $G_0 = 0$ and $H_0 = 4$ in Theorem 1.5,

$$G_{-n} = \frac{1}{6}(-u)^{-n}(-6G_{3n} + 6H_nG_{2n} - 3H_n^2G_n + 3H_{2n}G_n), \tag{1.15}$$

$$H_{-n} = \frac{1}{6}(-u)^{-n}\left(H_n^3 + 2H_{3n} - 3H_{2n}H_n\right), \tag{1.16}$$

respectively.

If we define the square matrix A of order 4 as

$$A = A_{rstu} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{1.17}$$

and also define

$$B_n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix} \tag{1.18}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} + uW_{n-2} & tW_n + uW_{n-1} & uW_n \\ W_n & sW_{n-1} + tW_{n-2} + uW_{n-3} & tW_{n-1} + uW_{n-2} & uW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} + uW_{n-4} & tW_{n-2} + uW_{n-3} & uW_{n-2} \\ W_{n-2} & sW_{n-3} + tW_{n-4} + uW_{n-5} & tW_{n-3} + uW_{n-4} & uW_{n-3} \end{pmatrix} \tag{1.19}$$

then we get the following Theorem.

Theorem 1.7. *[9, Theorem 19] For all integers m, n, we have*

(a) $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}. \tag{1.20}$$

(b) $U_1A^n = A^nU_1$.

(c) $U_{n+m} = U_nB_m = B_mU_n$.

Theorem 1.8. *[9, Theorem 20] For all integers m, n, we have*

$$W_{n+m} = W_nG_{m+1} + W_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + W_{n-2}(tG_m + uG_{m-1}) + uW_{n-3}G_m. \tag{1.21}$$

In the next sections, we present new results.

2 Generalized Friedrich Sequence

In this paper, we consider the case $r = 2, s = 0, t = 1, u = -2$. A generalized Friedrich sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n = 2W_{n-1} + W_{n-3} - 2W_{n-4} \tag{2.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = \frac{1}{2}W_{-(n-1)} + W_{-(n-3)} - \frac{1}{2}W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integers n.

Characteristic equation of $\{W_n\}$ is

$$z^4 - 2z^3 - z + 2 = (z^3 - z^2 - z - 2)(z - 1) = (z^2 + z + 1)(z - 2)(z - 1) = 0$$

whose roots are

$$\begin{aligned} \alpha &= 2, \\ \beta &= \frac{-1 + i\sqrt{3}}{2}, \\ \gamma &= \frac{-1 - i\sqrt{3}}{2}, \\ \delta &= 1. \end{aligned}$$

Note that

$$\begin{aligned} \alpha + \beta + \gamma + \delta &= 2, \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= 0, \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 1, \\ \alpha\beta\gamma\delta &= 2. \end{aligned}$$

Note also that

$$\begin{aligned} \alpha + \beta + \gamma &= 1, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\ \alpha\beta\gamma &= 2. \end{aligned}$$

The first few generalized Friedrich numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Friedrich numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{1}{2}(W_0+2W_2-W_3)$
2	W_2	$\frac{1}{4}(W_0+4W_1-W_3)$
3	W_3	$\frac{1}{8}(9W_0-W_3)$
4	$W_1-2W_0+2W_3$	$\frac{1}{16}(9W_0+16W_2-9W_3)$
5	$W_2-4W_0+4W_3$	$\frac{1}{32}(9W_0+32W_1-9W_3)$
6	$9W_3-8W_0$	$\frac{1}{64}(73W_0-9W_3)$
7	$W_1-18W_0+18W_3$	$\frac{1}{128}(73W_0+128W_2-73W_3)$
8	$W_2-36W_0+36W_3$	$\frac{1}{256}(73W_0+256W_1-73W_3)$
9	$73W_3-72W_0$	$\frac{1}{512}(585W_0-73W_3)$
10	$W_1-146W_0+146W_3$	$\frac{1}{1024}(585W_0+1024W_2-585W_3)$
11	$W_2-292W_0+292W_3$	$\frac{1}{2048}(585W_0+2048W_1-585W_3)$
12	$585W_3-584W_0$	$\frac{1}{4096}(4681W_0-585W_3)$
13	$W_1-1170W_0+1170W_3$	$\frac{1}{8192}(4681W_0+8192W_2-4681W_3)$

Note that the sequences $\{F_n\}$ and $\{C_n\}$ which are defined in the section Introduction, are the special cases of the generalized Friedrich sequence $\{W_n\}$. For convenience, we can give the definition of these two special cases of the sequence $\{W_n\}$ in this section as well. Friedrich sequence $\{F_n\}_{n\geq 0}$ and Friedrich-Lucas sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$F_n = 2F_{n-1}+F_{n-3}-2F_{n-4}, \quad F_0=0, F_1=1, F_2=2, F_3=4, \quad n\geq 4, \tag{2.2}$$

$$C_n = 2C_{n-1}+C_{n-3}-2C_{n-4}, \quad C_0=4, C_1=2, C_2=4, C_3=11, \quad n\geq 4. \tag{2.3}$$

The sequences $\{F_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$F_{-n} = \frac{1}{2}F_{-(n-1)}+F_{-(n-3)}-\frac{1}{2}F_{-(n-4)},$$

$$C_{-n} = \frac{1}{2}C_{-(n-1)}+C_{-(n-3)}-\frac{1}{2}C_{-(n-4)},$$

for $n=1,2,3,...$ respectively.

Next, we present the first few values of the Friedrich and Friedrich-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
F_n	0	1	2	4	9	18	36	73	146	292	585	1170	2340	4681
F_{-n}	0	0	0	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{9}{16}$	$-\frac{9}{32}$	$-\frac{9}{64}$	$-\frac{73}{128}$	$-\frac{73}{256}$	$-\frac{73}{512}$	$-\frac{585}{1024}$	$-\frac{585}{2048}$
C_n	4	2	4	11	16	32	67	128	256	515	1024	2048	4099	8192
C_{-n}	4	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{25}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{193}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1537}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$	$\frac{12289}{4096}$	$\frac{1}{8192}$

Theorem 1.1 can be used to obtain the Binet formula of generalized Friedrich numbers. Using these (the above) roots and the recurrence relation, Binet's formula of generalized Friedrich numbers can be given as follows:

Theorem 2.1. *(Four Distinct Roots Case: $\alpha\neq\beta\neq\gamma\neq\delta=1$) For all integers n, Binet's formula of generalized*

Friedrich numbers is

$$\begin{aligned} W_n &= \frac{(\alpha W_3 - \alpha(2-\alpha)W_2 + (-\alpha^2+\alpha+2)W_1 - 2W_0)\alpha^n}{2\alpha^2+5\alpha-4} \\ &+ \frac{(\beta W_3 - \beta(2-\beta)W_2 + (-\beta^2+\beta+2)W_1 - 2W_0)\beta^n}{2\beta^2+5\beta-4} \\ &+ \frac{(\gamma W_3 - \gamma(2-\gamma)W_2 + (-\gamma^2+\gamma+2)W_1 - 2W_0)\gamma^n}{2\gamma^2+5\gamma-4} \\ &+ \frac{W_3 - W_2 - W_1 - 2W_0}{-3}. \end{aligned}$$

Friedrich and Friedrich-Lucas numbers can be expressed using Binet's formulas as follows:

Corollary 2.2. *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers n, Binet's formulas of Friedrich and Friedrich-Lucas numbers are*

$$\begin{aligned} F_n &= \frac{(\alpha^2+\alpha+2)\alpha^n}{2\alpha^2+5\alpha-4} + \frac{(\beta^2+\beta+2)\beta^n}{2\beta^2+5\beta-4} + \frac{(\gamma^2+\gamma+2)\gamma^n}{2\gamma^2+5\gamma-4} - \frac{1}{3} \\ &= \frac{1}{7} \times 2^{n+2} - \frac{1}{42}(5+i\sqrt{3})\left(\frac{-1+i\sqrt{3}}{2}\right)^n - \frac{1}{42}(5-i\sqrt{3})\left(\frac{-1-i\sqrt{3}}{2}\right)^n - \frac{1}{3}, \end{aligned} \tag{2.4}$$

and

$$C_n = \alpha^n + \beta^n + \gamma^n + 1 = 2^n + \left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n + 1, \tag{2.5}$$

respectively.

Note that for all integers n, third-order Jacobsthal, modified third-order Jacobsthal and third-order Jacobsthal-Lucas numbers can be expressed using Binet's formulas as

$$J_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \tag{2.6}$$

$$K_n = \alpha^n + \beta^n + \gamma^n \tag{2.7}$$

$$j_n = \frac{(2\alpha^2-\alpha+2)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(2\beta^2-\beta+2)\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{(2\gamma^2-\gamma+2)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{2.8}$$

respectively, see Soykan [4] for more details. So, by using Binet's formulas of Friedrich, Friedrich-Lucas and third-order Jacobsthal, modified third-order Jacobsthal, third-order Jacobsthal-Lucas numbers, (or by using mathematical induction), we get the following Lemma which contains many identities:

Lemma 2.3. *For all integers n, the following equalities (identities) are true:*

(a)

- $J_{n+1} = F_{n+1} - F_n$.
- $2J_n = F_{n+3} - 2F_{n+2} + F_n$.
- $3F_{n+4} = 13J_{n+2} + 15J_{n+1} + 14J_n - 1$.
- $3F_n = J_{n+2} + 2J_n - 1$.
- $2J_n = -F_{n+2} + F_{n+1} + 3F_n + 1$.

(b)

- $147J_{n+3} = 40C_{n+3} + 21C_{n+2} - 7C_{n+1} - 54C_n$.

- $147J_n = -2C_{n+3} + 21C_{n+2} - 7C_{n+1} - 12C_n.$
- $C_{n+4} = 10J_{n+2} + 5J_{n+1} + 6J_n + 1.$
- $2C_n = -J_{n+2} + 7J_{n+1} - 3J_n + 2.$
- $147J_n = 19C_{n+2} - 9C_{n+1} - 16C_n + 6.$
- $C_{n+1} + 6C_n = 19J_{n+1} - 10J_n + 7.$

(c)

- $K_{n+3} = F_{n+3} + F_{n+2} + 4F_{n+1} - 6F_n.$
- $4K_n = -3F_{n+3} + 4F_{n+2} + 16F_{n+1} - 17F_n.$
- $147F_{n+4} = 195K_{n+2} + 181K_{n+1} + 202K_n - 49.$
- $147F_n = 17K_{n+2} + 10K_{n+1} - 4K_n - 49.$
- $4K_n = F_{n+2} + 13F_{n+1} - 23F_n - 3.$
- $3(17F_{n+1} - 27F_n) = 14K_n - K_{n+1} + 10.$

(d)

- $3K_{n+3} = 4C_{n+3} - C_{n+2} - C_{n+1} - 2C_n.$
- $3K_n = C_{n+3} - C_{n+2} - C_{n+1} + C_n.$
- $C_{n+4} = 2K_{n+2} + 3K_{n+1} + 2K_n + 1.$
- $C_n = K_n + 1.$
- $K_n = C_n - 1.$

(e)

- $j_{n+3} = F_{n+3} + 3F_{n+2} - 4F_n.$
- $j_n = -F_{n+3} + 3F_{n+2} - 2F_n.$
- $9F_{n+4} = 11j_{n+2} + 10j_n + 9j_{n+1} - 3.$
- $18F_n = j_{n+2} + 3j_{n+1} - j_n - 6.$
- $j_n = 2F_{n+2} - F_{n+1} - 4F_n - 1.$
- $3(F_{n+1} - 4F_n) = -2j_{n+1} + j_n + 3.$

(f)

- $49j_{n+3} = 72C_{n+3} - 21C_{n+2} + 7C_{n+1} - 58C_n.$
- $49j_n = 16C_{n+3} - 21C_{n+2} + 7C_{n+1} - 2C_n.$
- $3C_{n+4} = 4j_{n+2} + 8j_n + 9j_{n+1} + 3.$
- $24C_n = 11j_{n+2} - 21j_{n+1} + 19j_n + 24.$
- $49j_n = -5C_{n+2} + 23C_{n+1} + 30C_n - 48.$
- $11C_{n+1} + 10C_n = 5j_{n+1} + 18j_n + 21.$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n (by setting $r = 2, s = 0, t = 1, u = -2$ in Lemma 1.3).

Lemma 2.4. *Suppose that* $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ *is the ordinary generating function of the generalized Friedrich sequence* $\{W_n\}$. *Then,* $\sum_{n=0}^{\infty} W_n z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - 2W_0)z + (W_2 - 2W_1)z^2 + (W_3 - 2W_2 - W_0)z^3}{1 - 2z - z^3 + 2z^4}. \tag{2.9}$$

The previous lemma gives the following results as particular examples.

Corollary 2.5. *Generating functions of Friedrich and Friedrich-Lucas numbers are*

$$\sum_{n=0}^{\infty} F_n z^n = \frac{z}{1 - 2z - z^3 + 2z^4} = \frac{z}{(2z^3 + z^2 + z - 1)(z - 1)}, \tag{2.10}$$

$$\sum_{n=0}^{\infty} C_n z^n = \frac{4 - 6z - z^3}{1 - 2z - z^3 + 2z^4} = \frac{4 - 6z - z^3}{(2z^3 + z^2 + z - 1)(z - 1)}, \tag{2.11}$$

respectively.

3 Simson Formulas

Now, we present Simson's formula of generalized Friedrich numbers (by setting $r = 2, s = 0, t = 1, u = -2$ in Theorem 1.4).

Theorem 3.1 (Simson's Formula of Generalized Friedrich Numbers). *For all integers* n, *we have*
$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = 2^{n-3} \times (W_3 - W_0)(W_3 - W_2 - W_1 - 2W_0)(W_3^2 + 7W_2^2 + 7W_1^2 + 4W_0^2 - 5W_2W_3 + W_1W_3 - 7W_1W_2 + 2W_0W_3 - 2W_0W_2 - 8W_0W_1).$$

The previous theorem gives the following results as particular examples.

Corollary 3.2. *For all integers* n, *the Simson's formulas of Friedrich and Friedrich-Lucas numbers are given as*

$$\begin{vmatrix} F_{n+3} & F_{n+2} & F_{n+1} & F_n \\ F_{n+2} & F_{n+1} & F_n & F_{n-1} \\ F_{n+1} & F_n & F_{n-1} & F_{n-2} \\ F_n & F_{n-1} & F_{n-2} & F_{n-3} \end{vmatrix} = 2^{n-1},$$

$$\begin{vmatrix} C_{n+3} & C_{n+2} & C_{n+1} & C_n \\ C_{n+2} & C_{n+1} & C_n & C_{n-1} \\ C_{n+1} & C_n & C_{n-1} & C_{n-2} \\ C_n & C_{n-1} & C_{n-2} & C_{n-3} \end{vmatrix} = -1323 \times 2^{n-3},$$

respectively.

4 Some Identities

In this section, we obtain some identities of Friedrich and Friedrich-Lucas numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{F_n\}$.

Lemma 4.1. *The following equalities are true:*

(a) $16W_n = (9W_0 + 16W_2 - 9W_3)F_{n+5} - 16(2W_2 - W_3)F_{n+4} - 16(2W_0 - W_1)F_{n+3} - (9W_0 + 32W_1 - 9W_3)F_{n+2}$.

(b) $8W_n = (9W_0 - W_3)F_{n+4} - 8(2W_0 - W_1)F_{n+3} - 8(2W_1 - W_2)F_{n+2} - (9W_0 + 16W_2 - 9W_3)F_{n+1}$.

(c) $4W_n = (W_0 + 4W_1 - W_3)F_{n+3} - 4(2W_1 - W_2)F_{n+2} - 4(2W_2 - W_3)F_{n+1} - (9W_0 - W_3)F_n$.

(d) $2W_n = (W_0 + 2W_2 - W_3)F_{n+2} - 2(2W_2 - W_3)F_{n+1} - 2(2W_0 - W_1)F_n - (W_0 + 4W_1 - W_3)F_{n-1}$.

(e) $W_n = W_0F_{n+1} + (W_1 - 2W_0)F_n + (W_2 - 2W_1)F_{n-1} + (W_3 - 2W_2 - W_0)F_{n-2}$.

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$W_n = a \times F_{n+5} + b \times F_{n+4} + c \times F_{n+3} + d \times F_{n+2}$$

and solving the system of equations

$$\begin{aligned} W_0 &= a \times F_5 + b \times F_4 + c \times F_3 + d \times F_2 \\ W_1 &= a \times F_6 + b \times F_5 + c \times F_4 + d \times F_3 \\ W_2 &= a \times F_7 + b \times F_6 + c \times F_5 + d \times F_4 \\ W_3 &= a \times F_8 + b \times F_7 + c \times F_6 + d \times F_5 \end{aligned}$$

we find that $a = \frac{1}{16}(9W_0 + 16W_2 - 9W_3), b = W_3 - 2W_2, c = W_1 - 2W_0, d = \frac{1}{16}(9W_3 - 32W_1 - 9W_0)$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{W_n\}$ and $\{C_n\}$.

Lemma 4.2. *The following equalities are true:*

(a) $588W_n = -(139W_0 + 28W_1 + 112W_2 - 83W_3)C_{n+5} + 28(4W_0 + 7W_2 - 4W_3)C_{n+4} + 28(8W_0 - W_3)C_{n+3} + (195W_0 + 224W_1 + 112W_2 - 139W_3)C_{n+2}$.

(b) $294W_n = -(83W_0 + 28W_1 + 14W_2 - 27W_3)C_{n+4} + 14(8W_0 - W_3)C_{n+3} + 14(2W_0 + 7W_1 - 2W_3)C_{n+2} + (139W_0 + 28W_1 + 112W_2 - 83W_3)C_{n+1}$.

(c) $147W_n = -(27W_0 + 28W_1 + 14W_2 - 20W_3)C_{n+3} + 7(2W_0 + 7W_1 - 2W_3)C_{n+2} + 7(4W_0 + 7W_2 - 4W_3)C_{n+1} + (83W_0 + 28W_1 + 14W_2 - 27W_3)C_n$.

(d) $147W_n = -(40W_0 + 7W_1 + 28W_2 - 26W_3)C_{n+2} + 7(4W_0 + 7W_2 - 4W_3)C_{n+1} + 7(8W_0 - W_3)C_n + 2(27W_0 + 28W_1 + 14W_2 - 20W_3)C_{n-1}$.

(e) $147W_n = -(52W_0 + 14W_1 + 7W_2 - 24W_3)C_{n+1} + 7(8W_0 - W_3)C_n + 7(2W_0 + 7W_1 - 2W_3)C_{n-1} + 2(40W_0 + 7W_1 + 28W_2 - 26W_3)C_{n-2}$.

Now, we give a few basic relations between $\{F_n\}$ and $\{C_n\}$.

Lemma 4.3. *The following equalities are true:*

$$\begin{aligned} 147F_n &= 20C_{n+5} - 14C_{n+4} - 28C_{n+3} - 27C_{n+2}, \\ 147F_n &= 26C_{n+4} - 28C_{n+3} - 7C_{n+2} - 40C_{n+1}, \\ 147F_n &= 24C_{n+3} - 7C_{n+2} - 14C_{n+1} - 52C_n, \\ 147F_n &= 41C_{n+2} - 14C_{n+1} - 28C_n - 48C_{n-1}, \\ 147F_n &= 68C_{n+1} - 28C_n - 7C_{n-1} - 82C_{n-2}, \end{aligned}$$

and

$$\begin{aligned} 16C_n &= F_{n+5} + 48F_{n+4} - 96F_{n+3} - F_{n+2}, \\ 8C_n &= 25F_{n+4} - 48F_{n+3} - F_{n+1}, \\ 4C_n &= F_{n+3} + 12F_{n+1} - 25F_n, \\ 2C_n &= F_{n+2} + 6F_{n+1} - 12F_n - F_{n-1}, \\ C_n &= 4F_{n+1} - 6F_n - F_{n-2}. \end{aligned}$$

5 Relations Between Special Numbers

In this section, we present identities on Friedrich, Friedrich-Lucas numbers and third-order Jacobsthal, modified third-order Jacobsthal, third-order Jacobsthal-Lucas numbers. We know from Lemma 2.3 that

$$\begin{aligned} 3F_n &= J_{n+2}+2J_n-1, \\ C_n &= K_n+1. \end{aligned}$$

Note also that from Lemma 4.1 and Lemma 4.2, we have the formulas of W_n as

$$\begin{aligned} 4W_n &= (W_0+4W_1-W_3)F_{n+3}-4(2W_1-W_2)F_{n+2}-4(2W_2-W_3)F_{n+1}-(9W_0-W_3)F_n, \\ 147W_n &= -(27W_0+28W_1+14W_2-20W_3)C_{n+3}+7(2W_0+7W_1-2W_3)C_{n+2} \\ &\quad +7(4W_0+7W_2-4W_3)C_{n+1}+(83W_0+28W_1+14W_2-27W_3)C_n. \end{aligned}$$

Using the above identities, we obtain relation of generalized Friedrich numbers in the following forms (in terms of third-order Jacobsthal and modified third-order Jacobsthal numbers):

Lemma 5.1. *For all integers n, we have the following identities:*

(a) $6W_n = (-W_3+4W_2-2W_1-W_0)J_{n+2}+3(W_3-2W_2+W_0)J_{n+1}+(W_3-4W_2+8W_1-5W_0)J_n-2W_3+2W_2+2W_1+4W_0$.

(b) $147W_n = (6W_3-14W_2+21W_1-13W_0)K_{n+2}+(-8W_3+35W_2-28W_1+W_0)K_{n+1}+(13W_3-14W_2-28W_1+29W_0)K_n-49W_3+49W_2+49W_1+98W_0$.

6 On the Recurrence Properties of Generalized Friedrich Sequence

Taking $r=2, s=0, t=1, u=-2$ in Theorem 1.5, we obtain the following Proposition.

Proposition 6.1. *For $n\in\mathbb{Z}$, generalized Friedrich numbers (the case $r=2, s=0, t=1, u=-2$) have the following identity:*

$$W_{-n} = \frac{2^{-n-1}}{3}(-6W_{3n}+6C_nW_{2n}-3C_n^2W_n+3C_{2n}W_n+W_0C_n^3+2W_0C_{3n}-3W_0C_nC_{2n}).$$

From the above Proposition 6.1 (or by taking $G_n=F_n$ and $H_n=C_n$ in (1.15) and (1.16) respectively), we have the following corollary which gives the connection between the special cases of generalized Friedrich sequence at the positive index and the negative index: for Friedrich and Friedrich-Lucas numbers: take $W_n=F_n$ with $F_0=0, F_1=1, F_2=2, F_3=4$ and take $W_n=C_n$ with $C_0=4, C_1=2, C_2=4, C_3=11$, respectively. Note that in this case $H_n=C_n$.

Corollary 6.2. *For $n\in\mathbb{Z}$, we have the following recurrence relations:*

(a) *Friedrich sequence:*

$$F_{-n} = \frac{2^{-n-1}}{3}(-6F_{3n}+6C_nF_{2n}-3C_n^2F_n+3C_{2n}F_n).$$

(b) *Friedrich-Lucas sequence:*

$$C_{-n} = \frac{2^{-n-1}}{3}\left(C_n^3+2C_{3n}-3C_{2n}C_n\right).$$

We can also present the formulas of F_{-n} and C_{-n} in the following forms.

Corollary 6.3. *For $n\in\mathbb{Z}$, we have the following recurrence relations:*

(a) $F_{-n} = \frac{2^{-n-5}}{3}(-96F_{3n}+24(F_{n+3}+12F_{n+1}-25F_n)F_{2n}-3(F_{n+3}+12F_{n+1}-25F_n)^2F_n+12(F_{2n+3}+12F_{2n+1}-25F_{2n})F_n)$.

(b) $3F_{-n} = \dfrac{1}{2^n}(3J_n^2 + 6J_{n-2}^2 + (J_{n+2} - 7J_{n+1} + 2J_{n-2})J_n - 14J_{n-1}J_{n-2} + 2J_{2n} + 4J_{2n-4} - 2^n).$

(c) $147F_{-n} = \dfrac{1}{2^n}(-2K_n^2 + 10K_{n-1}^2 + 34K_{n-2}^2 + 2K_{2n} - 10K_{2n-2} - 34K_{2n-4} - 49 \times 2^n).$

(d) $2^{n+2}C_{-n} = -9J_n^2+42J_{n-1}^2-12J_{n-2}^2-(3J_{n+2}-21J_{n+1}+98J_{n-1}+4J_{n-2})J_n+14(J_{n+1}+2J_{n-2})J_{n-1}-6J_{2n}+28J_{2n-2} - 8J_{2n-4} + 2^{n+2}.$

(e) $C_{-n} = \dfrac{1}{2^{n+1}}(K_n^2 - K_{2n} + 2^{n+1}).$

Proof. We use the identities, see Soykan [17],

$$\begin{aligned} J_{-n} &= \frac{1}{2^{n+1}}(3J_n^2 + 2J_{2n} + J_{n+2}J_n - 7J_{n+1}J_n), \\ K_{-n} &= \frac{1}{2^{n+1}}(K_n^2 - K_{2n}). \end{aligned}$$

(a) By using the identity $4C_n = F_{n+3} + 12F_{n+1} - 25F_n$ and Corollary 6.2, (or by using Corollary 1.6 (a)), we obtain (a).

(b) Since

$$3F_n = J_{n+2} + 2J_n - 1,$$

and

$$J_{-n} = \frac{1}{2^{n+1}}(3J_n^2 + 2J_{2n} + J_{n+2}J_n - 7J_{n+1}J_n),$$

we get (b)

(c) Since $147F_n = 17K_{n+2} + 10K_{n+1} - 4K_n - 49$ and $K_{-n} = \frac{1}{2^{n+1}}(K_n^2 - K_{2n})$, we obtain (c).

(d) Since $2C_n = -J_{n+2} + 7J_{n+1} - 3J_n + 2$ and $J_{-n} = \frac{1}{2^{n+1}}(3J_n^2 + 2J_{2n} + J_{n+2}J_n - 7J_{n+1}J_n)$, , we get (d).

(e) Since $C_n = K_n + 1$ and $K_{-n} = \frac{1}{2^{n+1}}(K_n^2 - K_{2n})$, we obtain (e). □

7 Sum Formulas

The following Corollary gives sum formulas of third-order Jacobsthal numbers.

Corollary 7.1. *[4] For $n \geq 0$, third-order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n} J_k = \frac{1}{3}(J_{n+3} - J_{n+1} - 1).$

(b) $\sum_{k=0}^{n} J_{2k} = \frac{1}{3}(J_{2n+1} + 2J_{2n} - 1).$

(c) $\sum_{k=0}^{n} J_{2k+1} = \frac{1}{3}(J_{2n+2} + 2J_{2n+1}).$

The following Corollary presents sum formulas of Friedrich and Friedrich-Lucas numbers.

Corollary 7.2. *For $n \geq 0$, Friedrich and Friedrich-Lucas numbers have the following properties (in terms of third-order Jacobsthal numbers):*

(a)

(i) $\sum_{k=0}^{n} F_k = \frac{1}{3}(2J_{n+2} + J_{n+1} + 2J_n - n - 3).$

(ii) $\sum_{k=0}^{n} F_{2k} = \frac{1}{3}(J_{2n+2} + J_{2n+1} + 2J_{2n} - n - 2).$

(iii) $\sum_{k=0}^{n} F_{2k+1} = \frac{1}{3}(2J_{2n+2} + 3J_{2n+1} + 2J_{2n} - n - 2).$

(b)

(i) $\sum_{k=0}^{n} C_k = 3J_{n+1} + J_n + n + 1.$

(ii) $\sum_{k=0}^{n} C_{2k} = \frac{1}{3}(2J_{2n+2} + 5J_{2n+1} - 4J_{2n} + 3n + 5).$

(iii) $\sum_{k=0}^{n} C_{2k+1} = \frac{1}{3}(7J_{2n+2} - 2J_{2n+1} + 4J_{2n} + 3n + 1).$

Proof. The proof follows from Corollary 7.1 and the identities

$$\begin{aligned} 3F_n &= J_{n+2} + 2J_n - 1, \\ 2C_n &= -J_{n+2} + 7J_{n+1} - 3J_n + 2. \end{aligned} \qquad \square$$

8 Matrices and Identities Related With Generalized Friedrich Numbers

If we define the square matrix A of order 4 as

$$A = \begin{pmatrix} 2 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{8.1}$$

and also define

$$B_n = \begin{pmatrix} F_{n+1} & F_{n-1} - 2F_{n-2} & F_n - 2F_{n-1} & -2F_n \\ F_n & F_{n-2} - 2F_{n-3} & F_{n-1} - 2F_{n-2} & -2F_{n-1} \\ F_{n-1} & F_{n-3} - 2F_{n-4} & F_{n-2} - 2F_{n-3} & -2F_{n-2} \\ F_{n-2} & F_{n-4} - 2F_{n-5} & F_{n-3} - 2F_{n-4} & -2F_{n-3} \end{pmatrix} \tag{8.2}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & W_{n-1} - 2W_{n-2} & W_n - 2W_{n-1} & -2W_n \\ W_n & W_{n-2} - 2W_{n-3} & W_{n-1} - 2W_{n-2} & -2W_{n-1} \\ W_{n-1} & W_{n-3} - 2W_{n-4} & W_{n-2} - 2W_{n-3} & -2W_{n-2} \\ W_{n-2} & W_{n-4} - 2W_{n-5} & W_{n-3} - 2W_{n-4} & -2W_{n-3} \end{pmatrix}. \tag{8.3}$$

then we get the following Theorem (by setting $r = 2, s = 0, t = 1, u = -2$ in Theorem 1.7).

Theorem 8.1. *For all integers m, n, we have*

(a) $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} 2 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_{n-1} - 2F_{n-2} & F_n - 2F_{n-1} & -2F_n \\ F_n & F_{n-2} - 2F_{n-3} & F_{n-1} - 2F_{n-2} & -2F_{n-1} \\ F_{n-1} & F_{n-3} - 2F_{n-4} & F_{n-2} - 2F_{n-3} & -2F_{n-2} \\ F_{n-2} & F_{n-4} - 2F_{n-5} & F_{n-3} - 2F_{n-4} & -2F_{n-3} \end{pmatrix}. \tag{8.4}$$

(b) $U_1 A^n = A^n U_1$.

(c) $U_{n+m} = U_n B_m = B_m U_n$.

Using the above last Theorem and the identity

$$3F_n = J_{n+2} + 2J_n - 1,$$

we obtain the following identity for third-order Jacobsthal numbers.

Corollary 8.2. *For all integers n, we have the following formula for third-order Jacobsthal numbers:*

$$A^n = \begin{pmatrix} 2 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{3}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

where

$$\begin{aligned} a_{11} &= J_{n+3} + 2J_{n+1} - 1 \\ a_{21} &= J_{n+2} + 2J_n - 1 \\ a_{31} &= J_{n+1} + 2J_{n-1} - 1 \\ a_{41} &= J_n + 2J_{n-2} - 1 \end{aligned}$$

$$\begin{aligned} a_{12} &= J_{n+1} - 2J_n + 2J_{n-1} - 4J_{n-2} + 1 \\ a_{22} &= J_n - 2J_{n-1} + 2J_{n-2} - 4J_{n-3} + 1 \\ a_{32} &= J_{n-1} - 2J_{n-2} + 2J_{n-3} - 4J_{n-4} + 1 \\ a_{42} &= J_{n-2} - 2J_{n-3} + 2J_{n-4} - 4J_{n-5} + 1 \end{aligned}$$

$$\begin{aligned} a_{13} &= J_{n+2} - 2J_{n+1} + 2J_n - 4J_{n-1} + 1 \\ a_{23} &= J_{n+1} - 2J_n + 2J_{n-1} - 4J_{n-2} + 1 \\ a_{33} &= J_n - 2J_{n-1} + 2J_{n-2} - 4J_{n-3} + 1 \\ a_{43} &= J_{n-1} - 2J_{n-2} + 2J_{n-3} - 4J_{n-4} + 1 \end{aligned}$$

$$\begin{aligned} a_{14} &= -2(J_{n+2} + 2J_n - 1) \\ a_{24} &= -2(J_{n+1} + 2J_{n-1} - 1) \\ a_{34} &= -2(J_n + 2J_{n-2} - 1) \\ a_{44} &= -2(J_{n-1} + 2J_{n-3} - 1) \end{aligned}$$

Next, we present an identity for W_{n+m} (by setting $r = 2, s = 0, t = 1, u = -2$ in Theorem 1.8).

Theorem 8.3. *For all integers m, n, we have*

$$W_{n+m} = W_n F_{m+1} + W_{n-1}(F_{m-1} - 2F_{m-2}) + W_{n-2}(F_m - 2F_{m-1}) - 2W_{n-3}F_m \tag{8.5}$$

As particular cases of the above theorem, we give identities for F_{n+m} and C_{n+m}.

Corollary 8.4. *For all integers m, n, we have*

$$\begin{aligned} F_{n+m} &= F_n F_{m+1} + F_{n-1}(F_{m-1} - 2F_{m-2}) + F_{n-2}(F_m - 2F_{m-1}) - 2F_{n-3}F_m, & (8.6) \\ C_{n+m} &= C_n F_{m+1} + C_{n-1}(F_{m-1} - 2F_{m-2}) + C_{n-2}(F_m - 2F_{m-1}) - 2C_{n-3}F_m. & (8.7) \end{aligned}$$

9 Conclusions

Sequences have been fascinating topic for mathematicians for centuries. The Fibonacci and Lucas sequences are very well-known examples of second order recurrence sequences. For rich applications of these second order sequences in science and nature, one can see the citations in [18]. The generalization of Fibonacci sequence leads to several nice and interesting sequences.

As a fourth order sequence, we introduce the generalized Friedrich sequence (and it's two special cases, namely, Friedrich and Friedrich-Lucas sequences) and we present Binet's formulas, generating functions, Simson formulas, the sum formulas, some identities, recurrence properties and matrices for these sequences.

We have shown that there are close relations between Friedrich, Friedrich-Lucas numbers (which are fourth order linear recurences) and special third order linear recurences (numbers), namely third order Jacobsthal, modified third-order Jacobsthal, third order Jacobsthal-Lucas numbers.

Linear recurrence relations (sequences) have many applications. Next, we list applications of sequences which are linear recurrence relations.

First, we present some applications of second order sequences.

- For the applications of Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [19].
- For the application of Pell Numbers to the solutions of three-dimensional difference equation systems, see [20].
- For the application of Jacobsthal numbers to special matrices, see [21].
- For the application of generalized k-order Fibonacci numbers to hybrid quaternions, see [22].
- For the applications of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [23].
- For the applications of generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [24].
- For the applications of generalized Fibonacci numbers to binomial sums, see [25].
- For the application of generalized Jacobsthal numbers to hyperbolic numbers, see [26].
- For the application of generalized Fibonacci numbers to dual hyperbolic numbers, see [27].
- For the application of Laplace transform and various matrix operations to the characteristic polynomial of the Fibonacci numbers, see [28].
- For the application of Generalized Fibonacci Matrices to Cryptography, see [29].
- For the application of higher order Jacobsthal numbers to quaternions, see [30].
- For the application of Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [31].
- For the applications of Fibonacci numbers to lacunary statistical convergence, see [32].
- For the applications of Fibonacci numbers to lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [33].
- For the applications of Fibonacci numbers to ideal convergence on intuitionistic fuzzy normed linear spaces, see [34].

We now present some applications of third order sequences.

- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [35] and [36], respectively.
- For the application of Tribonacci numbers to special matrices, see [37].
- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see [38] and [39], respectively.
- For the application of Pell-Padovan numbers to groups, see [40].
- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [41].
- For the application of Gaussian Tribonacci numbers to various graphs, see [42].
- For the application of third-order Jacobsthal numbers to hyperbolic numbers, see [43].
- For the application of Narayan numbers to finite groups see [44].
- For the application of generalized third-order Jacobsthal sequence to binomial transform, see [45].
- For the application of generalized Generalized Padovan numbers to Binomial Transform, see [46].
- For the application of generalized Tribonacci numbers to Gaussian numbers, see [47].
- For the application of generalized Tribonacci numbers to Sedenions, see [48].

- For the application of Tribonacci and Tribonacci-Lucas numbers to matrices, see [49].
- For the application of generalized Tribonacci numbers to circulant matrix, see [50].
- For the application of Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see [51].

Next, we now list some applications of fourth order sequences.

- For the application of Tetranacci and Tetranacci-Lucas numbers to quaternions, see [52].
- For the application of generalized Tetranacci numbers to Gaussian numbers, see [53].
- For the application of Tetranacci and Tetranacci-Lucas numbers to matrices, see [54].
- For the application of generalized Tetranacci numbers to binomial transform, see [55].

We now present some applications of fifth order sequences.

- For the application of Pentanacci numbers to matrices, see [56].
- For the application of generalized Pentanacci numbers to quaternions, see [57].
- For the application of generalized Pentanacci numbers to binomial transform, see [58].

Disclaimer

This paper is an extended version of a preprint document of the same author. The preprint document is available in this link: https://www.researchgate.net/profile/Yueksel-Soykan/publication/365205545_Generalized_Friedrich-_Numbers/links/636af0e22f4bca7fd044ee62/Generalized-Friedrich-Numbers.pdf?origin=publication_detail
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Competing Interests

Author has declared that no competing interests exist.

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