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Minimum degree polynomial of graphs obtained by some graph operators

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Abstract: The minimum degree matrix $MD(G)$ of a graph G of order n is an $n \times n$ symmetric matrix whose $(i,j)^{th}$ entry is $min\{d_i, d_j\}$ whenever $i \neq j$, and zero otherwise, where d_i and d_j are the degrees of the i^{th} and j^{th} vertices of G, respectively. In the present work, we obtain the minimum degree polynomial of the graphs obtained by some graph operators (generalized xyz-point-line transformation graphs).

Keywords: Minimum degree matrix, minimum degree polynomial, eigenvalues, graph operators.

MSC: 05C07, 05C50.

1. Introduction

In the literature of graph theory, we can find several graph polynomials based on different matrices defined on the graph such as adjacency matrix [1], Laplacian matrix [2], signless Laplacian matrix [3,4], distance matrix [5], degree sum matrix [6,7], seidel matrix [8] etc. The purpose of this paper is to obtain the characteristic polynomial of the *minimum degree matrix* of a graph obtained by some graph operators (*generalized xyz-point-line transformation graphs*). For undefined graph theoretic terminologies and notions refer [1,9,10].

Let $G = (n, m)$ be a simple, undirected graph. Let $V(G)$ and $E(G)$ be the vertex set and edge set of G respectively. The *degree* $deg_G(v)$ (or $d_G(v)$) of a vertex $v \in V(G)$ is the number of edges incident to it in G. The graph G is *r-regular* if the degree of each vertex in G is r. Let $\{v_1, v_2, ..., v_n\}$ be the vertices of G and let $d_i = deg_G(v_i)$. The *minimum degree matrix* [11] of a graph G is an $n \times n$ matrix $MD(G) = [(md)_{ij}]$, whose elements are defined as

$$(md)_{ij} = \begin{cases} min\{d_i, d_j\} & if \;\; i \neq j, \\ 0 & otherwise. \end{cases}$$

Let I be the identity matrix and J be the matrix whose all entries are equal to 1. The *minimum degree polynomial* of a graph G is defined as

$$P_{MD(G)}(\xi) = det(\xi I - MD(G)).$$

The *eigenvalues* of the matrix $MD(G)$, denoted by $\xi_1, \xi_2, ..., \xi_n$ are called the *minimum degree eigenvalues* of G and their collection is called the *minimum degree spectra* of G. It is easy to see that if G is an *r-regular* graph, then $MD(G) = r(J - I)$. Therefore, for an *r-regular* graph G of order n,

$$P_{MD(G)}(\xi) = [\xi - r(n-1)][\xi + r]^{n-1}. \tag{1}$$

The *subdivision graph* [9] $S(G)$ of a graph G is a graph with the vertex set $V(S(G)) = V(G) \cup E(G)$ and two vertices of $S(G)$ are adjacent whenever they are incident in G. The *partial complement of subdivision graph* [12] $\overline{S}(G)$ of a graph G is a graph with the vertex set $V(\overline{S}(G)) = V(G) \cup E(G)$ and two vertices of $\overline{S}(G)$ are adjacent whenever they are nonincident in G.

In [13], Wu Bayoindureng *et al.* introduced the total transformation graphs and obtained the basic properties of total transformation graphs. For a graph $G = (V, E)$, let G^0 be the graph with $V(G^0) = V(G)$

and with no edges, G^1 the complete graph with $V(G^1) = V(G)$, $G^+ = G$, and $G^- = \overline{G}$. Let $\mathcal{G}$ denotes the set of simple graphs. The following graph operators depending on $x, y, z \in \{0, 1, +, -\}$ induce functions $T^{xyz} : \mathcal{G} \to \mathcal{G}$. These operators are introduced by Deng *et al.* in [14]. They referred these resulting graphs as xyz-transformations of G, denoted by $T^{xyz}(G) = G^{xyz}$ and obtained the Laplacian characteristic polynomials and some other Laplacian parameters of xyz-transformations of an r-regular graph G. Further, Basavanagoud [15] established the basic properties of these xyz-transformation graphs by calling them *xyz-point-line transformation graphs*.

Definition 1. [14] Given a graph G with vertex set $V(G)$ and edge set $E(G)$ and three variables $x, y, z \in \{0, 1, +, -\}$, the xyz-point-line transformation graph $T^{xyz}(G)$ of G is the graph with vertex set $V(T^{xyz}(G)) = V(G) \cup E(G)$ and the edge set $E(T^{xyz}(G)) = E((G)^x) \cup E((L(G))^y) \cup E(W)$ where $W = S(G)$ if $z = +$, $W = \overline{S}(G)$ if $z = -$, W is the graph with $V(W) = V(G) \cup E(G)$ and with no edges if $z = 0$ and W is the complete bipartite graph with parts $V(G)$ and $E(G)$ if $z = 1$.

Since there are 64 distinct 3-permutations of $\{0, 1, +, -\}$. Thus obtained 64 kinds of generalized *xyz-point-line transformation graphs*. There are 16 different graphs for each case when $z = 0$, $z = 1$, $z = +$, $z = -$.

For instance, the total graph $T(G)$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices of $T(G)$ are adjacent whenever they are adjacent or incident in G. The xyz-point-line transformation graph $T^{--+}(G)$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices of $T^{--+}(G)$ are adjacent whenever they are nonadjacent or incident in G.

The degree of vertices in the graphs $T^{xyz}(G)$ are given in the following Theorems 2 and 3, which are helpful in proving our results.

Theorem 2. *[16] Let G be a graph of order n, size m and let v be the point-vertex of T^{xyz} corresponding to a vertex v of G. Then*

1. $$d_{T^{xy0}}(v) = \begin{cases} 0 & if\ x = 0, y \in \{0, 1, +, -\}, \\ n-1 & if\ x = 1, y \in \{0, 1, +, -\}, \\ d_G(v) & if\ x = +, y \in \{0, 1, +, -\}, \\ n-1-d_G(v) & if\ x = -, y \in \{0, 1, +, -\}. \end{cases}$$
2. $$d_{T^{xy1}}(v) = \begin{cases} m & if\ x = 0, y \in \{0, 1, +, -\}, \\ n+m-1 & if\ x = 1, y \in \{0, 1, +, -\}, \\ m+d_G(v) & if\ x = +, y \in \{0, 1, +, -\}, \\ n+m-1-d_G(v) & if\ x = -, y \in \{0, 1, +, -\}. \end{cases}$$
3. $$d_{T^{xy+}}(v) = \begin{cases} d_G(v) & if\ x = 0, y \in \{0, 1, +, -\}, \\ n-1+d_G(v) & if\ x = 1, y \in \{0, 1, +, -\}, \\ 2d_G(v) & if\ x = +, y \in \{0, 1, +, -\}, \\ n-1 & if\ x = -, y \in \{0, 1, +, -\}. \end{cases}$$
4. $$d_{T^{xy-}}(v) = \begin{cases} m-d_G(v) & if\ x = 0, y \in \{0, 1, +, -\}, \\ n+m-1-d_G(v) & if\ x = 1, y \in \{0, 1, +, -\}, \\ m & if\ x = +, y \in \{0, 1, +, -\}, \\ n+m-1-2d_G(v) & if\ x = -, y \in \{0, 1, +, -\}. \end{cases}$$

Theorem 3. *[16] Let G be a graph of order n, size m and let e be the line-vertex of T^{xyz} corresponding to an edge e of G. Then*

1. $$d_{T^{xy0}}(e) = \begin{cases} 0 & if\ y = 0, x \in \{0, 1, +, -\}, \\ m-1 & if\ y = 1, x \in \{0, 1, +, -\}, \\ d_G(e) & if\ y = +, x \in \{0, 1, +, -\}, \\ m-1-d_G(e) & if\ y = -, x \in \{0, 1, +, -\}. \end{cases}$$
2. $$d_{T^{xy1}}(e) = \begin{cases} n & if\ y = 0, x \in \{0, 1, +, -\}, \\ n+m-1 & if\ y = 1, x \in \{0, 1, +, -\}, \\ n+d_G(e) & if\ y = +, x \in \{0, 1, +, -\}, \\ n+m-1-d_G(e) & if\ y = -, x \in \{0, 1, +, -\}. \end{cases}$$

3. $d_{T^{xy+}}(e) = \begin{cases} 2 & if\ y=0, x\in\{0,1,+,-\}, \\ m+1 & if\ y=1, x\in\{0,1,+,-\}, \\ 2+d_G(e) & if\ y=+, x\in\{0,1,+,-\}, \\ m+1-d_G(e) & if\ y=-, x\in\{0,1,+,-\}. \end{cases}$
4. $d_{T^{xy-}}(e) = \begin{cases} n-2 & if\ y=0, x\in\{0,1,+,-\}, \\ n+m-3 & if\ y=1, x\in\{0,1,+,-\}, \\ n-2+d_G(e) & if\ y=+, x\in\{0,1,+,-\}, \\ n+m-3-d_G(e) & if\ y=-, x\in\{0,1,+,-\}. \end{cases}$

2. Degree exponent polynomial of graphs obtained by graph operations

In this section we obtain the minimum degree polynomial of graphs obtained by some graph operators. We use the following lemma in order to prove the following theorems.

Lemma 4. *[17] If a, b, c and d are real numbers, then the determinant of the form*

$$\begin{vmatrix} (\xi+a)I_{n_1}-aJ_{n_1} & -cJ_{n_1\times n_2} \\ -dJ_{n_2\times n_1} & (\xi+b)I_{n_2}-bJ_{n_2} \end{vmatrix} \tag{2}$$

of order n_1+n_2 can be expressed in the simplified form as

$$(\xi+a)^{n_1-1}(\xi+b)^{n_2-1}\{[\xi-(n_1-1)a][\xi-(n_2-1)b]-n_1n_2cd\}.$$

Theorem 5. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{01+}(G))}(\xi) &= (\xi+m+1)^{m-1}(\xi+r)^{n-1}\{\xi^2-[(n-1)r+(m-1)(m+1)]\xi \\ &\quad +(n-1)r(m-1)(m+1)-min\{m+1,r\}^2mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{01+}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree r and the remaining m vertices are with degree $m+1$. Hence

$$MD(T^{01+}(G)) = \begin{bmatrix} r(J_n-I_n) & min\{r,m+1\}J_{n\times m} \\ min\{r,m+1\}J_{m\times n} & (m+1)(J_m-I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{01+}(G))}(\xi) &= |\xi I - MD(T^{01+}(G))| \\ &= \begin{vmatrix} (\xi+r)I_n-rJ_n & -min\{r,m+1\}J_{n\times m} \\ -min\{r,m+1\}J_{m\times n} & (\xi+m+1)I_m-(m+1)J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 6. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{0-+}(G))}(\xi) &= (\xi+r)^{n-1}(\xi+m+3-2r)^{m-1}\{\xi^2-[(n-1)r+(m-1)(m+3-2r)]\xi \\ &\quad +(n-1)(m-1)r(m+3-2r)-min\{r,m+3-2r\}^2mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{0-+}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree r and the remaining m vertices are with degree $m+3-2r$. Hence

$$MD(T^{0-+}(G)) = \begin{bmatrix} r(J_n-I_n) & min\{r,m+3-2r\}J_{n\times m} \\ min\{r,m+3-2r\}J_{m\times n} & (m+3-2r)(J_m-I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{0-+}(G))}(\xi) &= |\xi I - MD(T^{0-+}(G))| \\ &= \begin{vmatrix} (\xi+r)I_n - rJ_n & -min\{r, m+3-2r\}J_{n\times m} \\ -min\{r, m+3-2r\}J_{m\times n} & (\xi+m+3-2r)I_m - (m+3-2r)J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 7. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{10+}(G))}(\xi) &= (\xi+n-1+r)^{n-1}(\xi+2)^{m-1}\{\xi^2 - [(n-1)(n-1+r)+2(m-1)]\xi \\ &\quad +2(n-1)(m-1)(n-1+r) - min\{2, n-1+r\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{10+}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $n-1+r$ and the remaining m vertices are with degree 2. Hence

$$MD(T^{10+}(G)) = \begin{bmatrix} (n-1+r)(J_n - I_n) & min\{2, n-1+r\}J_{n\times m} \\ min\{2, n-1+r\}J_{m\times n} & 2(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{10+}(G))}(\xi) &= |\xi I - MD(T^{10+}(G))| \\ &= \begin{vmatrix} (\xi+n-1+r)I_n - (n-1+r)J_n & -min\{2, n-1+r\}J_{n\times m} \\ -min\{2, n-1+r\}J_{m\times n} & (\xi+2)I_m - 2J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 8. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{11+}(G))}(\xi) &= (\xi+n-1+r)^{n-1}(\xi+m+1)^{m-1}\{\xi^2 - [(n-1)(n-1+r)+(m-1)(m+1)]\xi \\ &\quad +(n-1)(m-1)(n-1+r)(m+1) - min\{n-1+r, m+1\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{11+}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $n-1+r$ and the remaining m vertices are with degree $m+1$. Hence

$$MD(T^{11+}(G)) = \begin{bmatrix} (n-1+r)(J_n - I_n) & min\{n-1+r, m+1\}J_{n\times m} \\ min\{n-1+r, m+1\}J_{m\times n} & (m+1)(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{11+}(G))}(\xi) &= |\xi I - MD(T^{11+}(G))| \\ &= \begin{vmatrix} (\xi+n-1+r)I_n - (n-1+r)J_n & -min\{n-1+r, m+1\}J_{n\times m} \\ -min\{n-1+r, m+1\}J_{m\times n} & (\xi+m+1)I_m - (m+1)J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 9. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{1++}(G))}(\xi) &= (\xi+n-1+r)^{n-1}(\xi+2r)^{m-1}\{\xi^2 - [(n-1)(n-1+r)+(m-1)2r]\xi \\ &\quad +(n-1)(m-1)(n-1+r)2r - min\{2r, n-1+r\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{1++}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $n-1+r$ and the remaining m vertices are with degree $2r$. Hence

$$MD(T^{1++}(G)) = \begin{bmatrix} (n-1+r)(J_n - I_n) & min\{2r, n-1+r\}J_{n\times m} \\ min\{2r, n-1+r\}J_{m\times n} & 2r(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{1++}(G))}(\xi) &= |\xi I - MD(T^{1++}(G))| \\ &= \begin{vmatrix} (\xi+n-1+r)I_n - (n-1+r)J_n & -min\{2r, n-1+r\}J_{n\times m} \\ -min\{2r, n-1+r\}J_{m\times n} & (\xi+2r)I_m - 2rJ_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 10. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{1-+}(G))}(\xi) &= (\xi+n-1+r)^{n-1}(\xi+m+3-2r)^{m-1}\{\xi^2 - [(n-1)(n-1+r) \\ &\quad +(m-1)(m+3-2r)]\xi + (n-1)(m-1)(n-1+r)(m+3-2r) \\ &\quad -min\{m+3-2r, n-1+r\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{1-+}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $R_1 = n-1+r$ and the remaining m vertices are with degree $R_2 = m+3-2r$. Hence

$$MD(T^{1-+}(G)) = \begin{bmatrix} R_1(J_n - I_n) & min\{R_1, R_2\}J_{n\times m} \\ min\{R_1, R_2\}J_{m\times n} & R_2(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{1-+}(G))}(\xi) &= |\xi I - MD(T^{1-+}(G))| \\ &= \begin{vmatrix} (\xi+R_1)I_n - R_1J_n & -min\{R_1, R_2\}J_{n\times m} \\ -min\{R_1, R_2\}J_{m\times n} & (\xi+R_2)I_m - R_2J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 11. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{+1+}(G))}(\xi) &= (\xi+2r)^{n-1}(\xi+m+1)^{m-1}\{\xi^2 - [(n-1)2r + (m-1)(m+1)]\xi \\ &\quad +(n-1)(m-1)2r(m+1) - min\{2r, m+1\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{+1+}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $2r$ and the remaining m vertices are with degree $m+1$. Hence

$$MD(T^{+1+}(G)) = \begin{bmatrix} 2r(J_n - I_n) & min\{2r, m+1\}J_{n\times m} \\ min\{2r, m+1\}J_{m\times n} & (m+1)(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{+1+}(G))}(\xi) &= |\xi I - MD(T^{+1+}(G))| \\ &= \begin{vmatrix} (\xi+2r)I_n - 2rJ_n & -min\{2r, m+1\}J_{n\times m} \\ -min\{2r, m+1\}J_{m\times n} & (\xi+m+1)I_m - (m+1)J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 12. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{+-+}(G))}(\xi) &= (\xi+2r)^{n-1}(\xi+m+3-2r)^{m-1}\{\xi^2-[(n-1)2r+(m-1)(m+3-2r)]\xi \\ &\quad +(n-1)(m-1)2r(m+3-2r)-min\{2r,m+3-2r\}^2mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{+-+}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $2r$ and the remaining m vertices are with degree $m+3-2r$. Hence

$$MD(T^{+-+}(G)) = \begin{bmatrix} 2r(J_n-I_n) & min\{2r,m+3-2r\}J_{n\times m} \\ min\{2r,m+3-2r\}J_{m\times n} & (m+3-2r)(J_m-I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{+-+}(G))}(\xi) &= |\xi I - MD(T^{+-+}(G))| \\ &= \begin{vmatrix} (\xi+2r)I_n-2rJ_n & -min\{2r,m+3-2r\}J_{n\times m} \\ -min\{2r,m+3-2r\}J_{m\times n} & (\xi+m+3-2r)I_m-(m+3-2r)J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 13. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{-0+}(G))}(\xi) &= (\xi+n-1)^{n-1}(\xi+2)^{m-1}\{\xi^2-[(n-1)(n-1)+2(m-1)]\xi \\ &\quad +2(n-1)(m-1)(n-1)-min\{n-1,2\}^2mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{-0+}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $n-1$ and the remaining m vertices are with degree 2. Hence

$$MD(T^{-0+}(G)) = \begin{bmatrix} (n-1)(J_n-I_n) & min\{n-1,2\}J_{n\times m} \\ min\{n-1,2\}J_{m\times n} & 2(J_m-I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{-0+}(G))}(\xi) &= |\xi I - MD(T^{-0+}(G))| \\ &= \begin{vmatrix} (\xi+n-1)I_n-(n-1)J_n & -min\{n-1,2\}J_{n\times m} \\ -min\{n-1,2\}J_{m\times n} & (\xi+2)I_m-2J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 14. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{-1+}(G))}(\xi) &= (\xi+n-1)^{n-1}(\xi+m+1)^{m-1}\{\xi^2-[(n-1)(n-1)+(m-1)(m+1)]\xi \\ &\quad +(n-1)(m-1)(n-1)(m+1)-min\{n-1,m+1\}^2mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{-1+}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $n-1$ and the remaining m vertices are with degree $m+1$. Hence

$$MD(T^{-1+}(G)) = \begin{bmatrix} (n-1)(J_n-I_n) & min\{n-1,m+1\}J_{n\times m} \\ min\{n-1,m+1\}J_{m\times n} & (m+1)(J_m-I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{-1+}(G))}(\xi) &= |\xi I - MD(T^{-1+}(G))| \\ &= \begin{vmatrix} (\xi+n-1)I_n-(n-1)J_n & -min\{n-1,m+1\}J_{n\times m} \\ -min\{n-1,m+1\}J_{m\times n} & (\xi+m+1)I_m-(m+1)J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 15. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{-++}(G))}(\xi) &= (\xi+n-1)^{n-1}(\xi+2r)^{m-1}\{\xi^2-[(n-1)(n-1)+(m-1)2r]\xi \\ &\quad +(n-1)(m-1)(n-1)2r-min\{n-1,2r\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{-++}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $n-1$ and the remaining m vertices are with degree $2r$. Hence

$$MD(T^{-++}(G)) = \begin{bmatrix} (n-1)(J_n-I_n) & min\{n-1,2r\}J_{n\times m} \\ min\{n-1,2r\}J_{m\times n} & 2r(J_m-I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{-++}(G))}(\xi) &= |\xi I - MD(T^{-++}(G))| \\ &= \begin{vmatrix} (\xi+n-1)I_n-(n-1)J_n & -min\{n-1,2r\}J_{n\times m} \\ -min\{n-1,2r\}J_{m\times n} & (\xi+2r)I_m-2rJ_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 16. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{--+}(G))}(\xi) &= (\xi+n-1)^{n-1}(\xi+m+3-2r)^{m-1}\{\xi^2-[(n-1)(n-1) \\ &\quad +(m-1)(m+3-2r)]\xi+(n-1)(m-1)(n-1)(m+3-2r) \\ &\quad -min\{n-1,m+3-2r\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{--+}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $n-1$ and the remaining m vertices are with degree $R=m+3-2r$. Hence

$$MD(T^{--+}(G)) = \begin{bmatrix} (n-1)(J_n-I_n) & min\{n-1,R\}J_{n\times m} \\ min\{n-1,R\}J_{m\times n} & R(J_m-I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{--+}(G))}(\xi) &= |\xi I - MD(T^{--+}(G))| \\ &= \begin{vmatrix} (\xi+n-1)I_n-(n-1)J_n & -min\{n-1,R\}J_{n\times m} \\ -min\{n-1,R\}J_{m\times n} & (\xi+R)I_m-RJ_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 17. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{00-}(G))}(\xi) &= (\xi+m-r)^{n-1}(\xi+n-2)^{m-1}\{\xi^2-[(n-1)(m-r)+(m-1)(n-2)]\xi \\ &\quad +(n-1)(m-1)(m-r)(n-2)-min\{m-r,n-2\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{00-}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $m-r$ and the remaining m vertices are with degree $n-2$. Hence

$$MD(T^{00-}(G)) = \begin{bmatrix} (m-r)(J_n - I_n) & min\{m-r, n-2\}J_{n\times m} \\ min\{m-r, n-2\}J_{m\times n} & (n-2)(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{00-}(G))}(\xi) &= |\xi I - MD(T^{00-}(G))| \\ &= \begin{vmatrix} (\xi+m-r)I_n - (m-r)J_n & -min\{m-r, n-2\}J_{n\times m} \\ -min\{m-r, n-2\}J_{m\times n} & (\xi+n-2)I_m - (n-2)J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 18. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{01-}(G))}(\xi) &= (\xi+m-r)^{n-1}(\xi+n+m-3)^{m-1}\{\xi^2 - [(n-1)(m-r)+(m-1)(n+m-3)]\xi \\ &\quad +(n-1)(m-1)(m-r)(n+m-3) - min\{m-r, n+m-3\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{01-}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $m-r$ and the remaining m vertices are with degree $R = n+m-3$. Hence

$$MD(T^{01-}(G)) = \begin{bmatrix} (m-r)(J_n - I_n) & min\{m-r, R\}J_{n\times m} \\ min\{m-r, R\}J_{m\times n} & R(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{01-}(G))}(\xi) &= |\xi I - MD(T^{01-}(G))| \\ &= \begin{vmatrix} (\xi+m-r)I_n - (m-r)J_n & -min\{m-r, R\}J_{n\times m} \\ -min\{m-r, R\}J_{m\times n} & (\xi+R)I_m - RJ_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 19. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{0+-}(G))}(\xi) &= (\xi+m-r)^{n-1}(\xi+n-4+2r)^{m-1}\{\xi^2 - [(n-1)(m-r)+(m-1)(n-4+2r)]\xi \\ &\quad +(n-1)(m-1)(m-r)(n-4+2r) - min\{m-r, n-4+2r\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{0+-}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $m-r$ and the remaining m vertices are with degree $R = n-4+2r$. Hence

$$MD(T^{0+-}(G)) = \begin{bmatrix} (m-r)(J_n - I_n) & min\{m-r, R\}J_{n\times m} \\ min\{m-r, R\}J_{m\times n} & R(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{0+-}(G))}(\xi) &= |\xi I - MD(T^{0+-}(G))| \\ &= \begin{vmatrix} (\xi+m-r)I_n - (m-r)J_n & -min\{m-r, R\}J_{n\times m} \\ -min\{m-r, R\}J_{m\times n} & (\xi+R)I_m - RJ_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 20. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{0--}(G))}(\xi) &= (\xi+m-r)^{n-1}(\xi+n+m-1-2r)^{m-1}\{\xi^2-[(n-1)(m-r)\\ &\quad +(m-1)(n+m-1-2r)]\xi+(n-1)(m-1)(m-r)(n+m-1-2r)\\ &\quad -min\{m-r,n+m-1-2r\}^2mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{0--}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $m-r$ and the remaining m vertices are with degree $R = n+m-1-2r$. Hence

$$MD(T^{0--}(G)) = \begin{bmatrix} (m-r)(J_n-I_n) & min\{m-r,R\}J_{n\times m} \\ min\{m-r,R\}J_{m\times n} & R(J_m-I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{0--}(G))}(\xi) &= |\xi I - MD(T^{0--}(G))|\\ &= \begin{vmatrix} (\xi+m-r)I_n-(m-r)J_n & -min\{m-r,R\}J_{n\times m} \\ -min\{m-r,R\}J_{m\times n} & (\xi+R)I_m-RJ_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 21. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{10-}(G))}(\xi) &= (\xi+n+m-r-1)^{n-1}(\xi+n-2)^{m-1}\{\xi^2-[(n-1)(n+m-r-1)+(m-1)(n-2)]\xi\\ &\quad +(n-1)(m-1)(n+m-r-1)(n-2)-min\{n+m-r-1,n-2\}^2mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{10-}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $R_1 = n+m-r-1$ and the remaining m vertices are with degree $R_2 = n+m-1-2r$. Hence

$$MD(T^{10-}(G)) = \begin{bmatrix} R_1(J_n-I_n) & min\{R_1,R_2\}J_{n\times m} \\ min\{R_1,R_2\}J_{m\times n} & R_2(J_m-I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{10-}(G))}(\xi) &= |\xi I - MD(T^{10-}(G))|\\ &= \begin{vmatrix} (\xi+R_1)I_n-R_1J_n & -min\{R_1,R_2\}J_{n\times m} \\ -min\{R_1,R_2\}J_{m\times n} & (\xi+R_2)I_m-R_2J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 22. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{11-}(G))}(\xi) &= (\xi+n+m-r-1)^{n-1}(\xi+n+m-3)^{m-1}\{\xi^2-[(n-1)(n+m-r-1)\\ &\quad +(m-1)(n+m-3)]\xi+(n-1)(m-1)(n+m-r-1)(n+m-3)\\ &\quad -min\{n+m-r-1,n+m-3\}^2mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{11-}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $R_1 = n+m-r-1$ and the remaining m vertices are with degree $R_2 = n+m-3$. Hence

$$MD(T^{11-}(G)) = \begin{bmatrix} R_1(J_n-I_n) & min\{R_1,R_2\}J_{n\times m} \\ min\{R_1,R_2\}J_{m\times n} & R_2(J_m-I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{11-}(G)}(\xi) &= |\xi I - MD(T^{11-}(G))| \\ &= \left| \begin{array}{cc} (\xi + R_1)I_n - R_1 J_n & -min\{R_1, R_2\} J_{n\times m} \\ -min\{R_1, R_2\} J_{m\times n} & (\xi + R_2)I_m - R_2 J_m \end{array} \right|. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 23. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{1+-}(G))}(\xi) &= (\xi + n + m - r - 1)^{n-1}(\xi + n + 2r - 4)^{m-1}\{\xi^2 - [(n-1)(n+m-r-1) \\ &\quad +(m-1)(n+2r-4)]\xi + (n-1)(m-1)(n+m-r-1)(n+2r-4) \\ &\quad -min\{n+m-r-1, n+2r-4\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{1+-}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $R_1 = n + m - r - 1$ and the remaining m vertices are with degree $R_2 = n + 2r - 4$. Hence

$$MD(T^{1+-}(G)) = \left[\begin{array}{cc} R_1(J_n - I_n) & min\{R_1, R_2\} J_{n\times m} \\ min\{R_1, R_2\} J_{m\times n} & R_2(J_m - I_m) \end{array} \right].$$

Therefore,

$$\begin{aligned} P_{MD(T^{1+-}(G))}(\xi) &= |\xi I - MD(T^{1+-}(G))| \\ &= \left| \begin{array}{cc} (\xi + R_1)I_n - R_1 J_n & -min\{R_1, R_2\} J_{n\times m} \\ -min\{R_1, R_2\} J_{m\times n} & (\xi + R_2)I_m - R_2 J_m \end{array} \right|. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 24. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{1--}(G))}(\xi) &= (\xi + n + m - r - 1)^{n-1}(\xi + n + m - 2r - 1)^{m-1}\{\xi^2 - [(n-1)(n+m-r-1) \\ &\quad +(m-1)(n+m-2r-1)]\xi + (n-1)(m-1)(n+m-r-1)(n+m-2r-1) \\ &\quad -min\{n+m-r-1, n+m-2r-1\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{1--}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $R_1 = n + m - r - 1$ and the remaining m vertices are with degree $R_2 = n + m - 2r - 1$. Hence

$$MD(T^{1--}(G)) = \left[\begin{array}{cc} R_1(J_n - I_n) & min\{R_1, R_2\} J_{n\times m} \\ min\{R_1, R_2\} J_{m\times n} & R_2(J_m - I_m) \end{array} \right].$$

Therefore,

$$\begin{aligned} P_{MD(T^{1--}(G))}(\xi) &= |\xi I - MD(T^{1--}(G))| \\ &= \left| \begin{array}{cc} (\xi + R_1)I_n - R_1 J_n & -min\{R_1, R_2\} J_{n\times m} \\ -min\{R_1, R_2\} J_{m\times n} & (\xi + R_2)I_m - R_2 J_m \end{array} \right|. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 25. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{+0-}(G))}(\xi) &= (\xi+m)^{n-1}(\xi+n-2)^{m-1}\{\xi^2-[(n-1)m+(m-1)(n-2)]\xi \\ &\quad +(n-1)(m-1)m(n-2)-min\{m,n-2\}^2mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{+0-}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree m and the remaining m vertices are with degree $n-2$. Hence

$$MD(T^{+0-}(G)) = \begin{bmatrix} m(J_n-I_n) & min\{m,n-2\}J_{n\times m} \\ min\{m,n-2\}J_{m\times n} & (n-2)(J_m-I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{+0-}(G))}(\xi) &= |\xi I - MD(T^{+0-}(G))| \\ &= \begin{vmatrix} (\xi+m)I_n - mJ_n & -min\{m,n-2\}J_{n\times m} \\ -min\{m,n-2\}J_{m\times n} & (\xi+n-2)I_m-(n-2)J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 26. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{+1-}(G))}(\xi) &= (\xi+m)^{n-1}(\xi+n+m-3)^{m-1}\{\xi^2-[(n-1)m+(m-1)(n+m-3)]\xi \\ &\quad +(n-1)(m-1)m(n+m-3)-min\{m,n+m-3\}^2mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{+1-}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree m and the remaining m vertices are with degree $n+m-3$. Hence

$$MD(T^{+1-}(G)) = \begin{bmatrix} m(J_n-I_n) & min\{m,n+m-3\}J_{n\times m} \\ min\{m,n+m-3\}J_{m\times n} & (n+m-3)(J_m-I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{+1-}(G))}(\xi) &= |\xi I - MD(T^{+1-}(G))| \\ &= \begin{vmatrix} (\xi+m)I_n - mJ_n & -min\{m,n+m-3\}J_{n\times m} \\ -min\{m,n+m-3\}J_{m\times n} & (\xi+n+m-3)I_m-(n+m-3)J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 27. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{++-}(G))}(\xi) &= (\xi+m)^{n-1}(\xi+n+2r-4)^{m-1}\{\xi^2-[(n-1)m+(m-1)(n+2r-4)]\xi \\ &\quad +(n-1)(m-1)m(n+2r-4)-min\{m,n+2r-4\}^2mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{++-}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree m and the remaining m vertices are with degree $n+2r-4$. Hence

$$MD(T^{++-}(G)) = \begin{bmatrix} m(J_n-I_n) & min\{m,n+2r-4\}J_{n\times m} \\ min\{m,n+2r-4\}J_{m\times n} & (n+2r-4)(J_m-I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{++-}(G))}(\xi) &= |\xi I - MD(T^{++-}(G))| \\ &= \begin{vmatrix} (\xi+m)I_n - mJ_n & -min\{m, n+2r-4\}J_{n\times m} \\ -min\{m, n+2r-4\}J_{m\times n} & (\xi+n+2r-4)I_m - (n+2r-4)J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 28. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{+--}(G))}(\xi) &= (\xi+m)^{n-1}(\xi+n+m-1-2r)^{m-1}\{\xi^2 - [(n-1)m + (m-1)(n+m-1-2r)]\xi \\ &\quad +(n-1)(m-1)m(n+m-1-2r) - min\{n+m-1-2r, n+m-1-2r\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{+--}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree m and the remaining m vertices are with degree $R = n+m-1-2r$. Hence

$$MD(T^{+--}(G)) = \begin{bmatrix} m(J_n - I_n) & min\{m, R\}J_{n\times m} \\ min\{m, R\}J_{m\times n} & R(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{+--}(G))}(\xi) &= |\xi I - MD(T^{+--}(G))| \\ &= \begin{vmatrix} (\xi+m)I_n - mJ_n & -min\{m, R\}J_{n\times m} \\ -min\{m, R\}J_{m\times n} & (\xi+R)I_m - RJ_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 29. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{-0-}(G))}(\xi) &= (\xi+n+m+3-4r)^{n-1}(\xi+n-2)^{m-1}\{\xi^2 - [(n-1)(n+m+3-4r) \\ &\quad +(m-1)(n-2)]\xi + (n-1)(m-1)(n+m+3-4r)(n-2) \\ &\quad -min\{n+m+3-4r, n-2\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{-0-}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $R = n+m+3-4r$ and the remaining m vertices are with degree $n-2$. Hence

$$MD(T^{-0-}(G)) = \begin{bmatrix} R(J_n - I_n) & min\{n-2, R\}J_{n\times m} \\ min\{n-2, R\}J_{m\times n} & (n-2)(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{-0-}(G))}(\xi) &= |\xi I - MD(T^{-0-}(G))| \\ &= \begin{vmatrix} (\xi+R)I_n - RJ_n & -min\{n-2, R\}J_{n\times m} \\ -min\{n-2, R\}J_{m\times n} & (\xi+n-2)I_m - (n-2)J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 30. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{-1-}(G))}(\xi) &= (\xi+n+m+3-4r)^{n-1}(\xi+n+m-3)^{m-1}\{\xi^2 - [(n-1)(n+m+3-4r) \\ &\quad +(m-1)(n+m-3)]\xi + (n-1)(m-1)(n+m+3-4r)(n+m-3) \\ &\quad -min\{n+m+3-4r, n+m-3\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{-1-}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $R = n+m+3-4r$ and the remaining m vertices are with degree $n+m-3$. Hence

$$MD(T^{-1-}(G)) = \begin{bmatrix} R(J_n - I_n) & min\{n+m-3, R\}J_{n\times m} \\ min\{n+m-3, R\}J_{m\times n} & (n+m-3)(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{-1-}(G))}(\xi) &= |\xi I - MD(T^{-1-}(G))| \\ &= \begin{vmatrix} (\xi+R)I_n - RJ_n & -min\{n+m-3, R\}J_{n\times m} \\ -min\{n+m-3, R\}J_{m\times n} & (\xi+n+m-3)I_m - (n+m-3)J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 31. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{-+-}(G))}(\xi) &= (\xi+n+m+3-4r)^{n-1}(\xi+n+2r-4)^{m-1}\{\xi^2 - [(n-1)(n+m+3-4r) \\ &\quad +(m-1)(n+2r-4)]\xi + (n-1)(m-1)(n+m+3-4r)(n+2r-4) \\ &\quad -min\{n+m+3-4r, n+2r-4\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{-+-}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $R = n+m+3-4r$ and the remaining m vertices are with degree $n+2r-4$. Hence

$$MD(T^{-+-}(G)) = \begin{bmatrix} R(J_n - I_n) & min\{n+2r-4, R\}J_{n\times m} \\ min\{n+2r-4, R\}J_{m\times n} & (n+2r-4)(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{-+-}(G))}(\xi) &= |\xi I - MD(T^{-+-}(G))| \\ &= \begin{vmatrix} (\xi+R)I_n - RJ_n & -min\{n+2r-4, R\}J_{n\times m} \\ -min\{n+2r-4, R\}J_{m\times n} & (\xi+n+2r-4)I_m - (n+2r-4)J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 32. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{---}(G))}(\xi) &= (\xi+n+m+3-4r)^{n-1}(\xi+n+m-2r-1)^{m-1}\{\xi^2 - [(n-1)(n+m+3-4r) \\ &\quad +(m-1)(n+m-2r-1)]\xi + (n-1)(m-1)(n+m+3-4r)(n+m-2r-1) \\ &\quad -min\{n+m+3-4r, n+m-2r-1\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{---}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $R_1 = n+m+3-4r$ and the remaining m vertices are with degree $R_2 = n+m-2r-1$. Hence

$$MD(T^{---}(G)) = \begin{bmatrix} R_1(J_n - I_n) & min\{R_1, R_2\}J_{n\times m} \\ min\{R_1, R_2\}J_{m\times n} & R_2(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{---}(G))}(\xi) &= |\xi I - MD(T^{---}(G))| \\ &= \begin{vmatrix} (\xi + R_1)I_n - R_1 J_n & -min\{R_1, R_2\}J_{n\times m} \\ -min\{R_1, R_2\}J_{m\times n} & (\xi + R_2)I_m - R_2 J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 33. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{001}(G))}(\xi) &= (\xi + m)^{n-1}(\xi + n)^{m-1}\{\xi^2 - [(n-1)m + (m-1)n]\xi \\ &\quad +(n-1)(m-1)mn - min\{m, n\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{001}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree m and the remaining m vertices are with degree n. Hence

$$MD(T^{001}(G)) = \begin{bmatrix} m(J_n - I_n) & min\{m, n\}J_{n\times m} \\ min\{m, n\}J_{m\times n} & n(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{001}(G))}(\xi) &= |\xi I - MD(T^{001}(G))| \\ &= \begin{vmatrix} (\xi + m)I_n - mJ_n & -min\{m, n\}J_{n\times m} \\ -min\{m, n\}J_{m\times n} & (\xi + n)I_m - nJ_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 34. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{011}(G))}(\xi) &= (\xi + m)^{n-1}(\xi + n + m - 3)^{m-1}\{\xi^2 - [(n-1)m + (m-1)(n+m-3)]\xi \\ &\quad +(n-1)(m-1)m(n+m-3) - min\{m, n+m-3\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{011}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree m and the remaining m vertices are with degree $n + m - 3$. Hence

$$MD(T^{011}(G)) = \begin{bmatrix} m(J_n - I_n) & min\{m, n+m-3\}J_{n\times m} \\ min\{m, n+m-3\}J_{m\times n} & (n+m-3)(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{011}(G))}(\xi) &= |\xi I - MD(T^{011}(G))| \\ &= \begin{vmatrix} (\xi + m)I_n - mJ_n & -min\{m, n+m-3\}J_{n\times m} \\ -min\{m, n+m-3\}J_{m\times n} & (\xi + n + m - 3)I_m - (n+m-3)J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 35. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{0+1}(G))}(\xi) &= (\xi + m)^{n-1}(\xi + n + 2r - 2)^{m-1}\{\xi^2 - [(n-1)m + (m-1)(n+2r-2)]\xi \\ &\quad +(n-1)(m-1)m(n+2r-2) - min\{m, n+2r-2\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{0+1}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree m and the remaining m vertices are with degree $n+2r-2$. Hence

$$MD(T^{0+1}(G)) = \begin{bmatrix} m(J_n - I_n) & min\{m, n+2r-2\}J_{n\times m} \\ min\{m, n+2r-2\}J_{m\times n} & (n+2r-2)(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{0+1}(G))}(\xi) &= |\xi I - MD(T^{0+1}(G))| \\ &= \begin{vmatrix} (\xi+m)I_n - mJ_n & -min\{m, n+2r-2\}J_{n\times m} \\ -min\{m, n+2r-2\}J_{m\times n} & (\xi+n+2r-2)I_m - (n+2r-2)J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 36. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{0-1}(G))}(\xi) &= (\xi+m)^{n-1}(\xi+n+m+1-2r)^{m-1}\{\xi^2 - [(n-1)m + (m-1)(n+m+1-2r)]\xi \\ &\quad +(n-1)(m-1)m(n+m+1-2r) - min\{m, n+m+1-2r\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{0-1}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree m and the remaining m vertices are with degree $R = n+m+1-2r$. Hence

$$MD(T^{0-1}(G)) = \begin{bmatrix} m(J_n - I_n) & min\{m, R\}J_{n\times m} \\ min\{m, R\}J_{m\times n} & R(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{0-1}(G))}(\xi) &= |\xi I - MD(T^{0-1}(G))| \\ &= \begin{vmatrix} (\xi+m)I_n - mJ_n & -min\{m, R\}J_{n\times m} \\ -min\{m, R\}J_{m\times n} & (\xi+R)I_m - RJ_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 37. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{101}(G))}(\xi) &= (\xi+n+m-1)^{n-1}(\xi+n)^{m-1}\{\xi^2 - [(n-1)(n+m-1) \\ &\quad +(m-1)n]\xi + (n-1)(m-1)(n+m-1)n - min\{n+m-1, n\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{101}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $n+m-1$ and the remaining m vertices are with degree n. Hence

$$MD(T^{101}(G)) = \begin{bmatrix} (n+m-1)(J_n - I_n) & min\{n+m-1, n\}J_{n\times m} \\ min\{n+m-1, n\}J_{m\times n} & n(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{101}(G))}(\xi) &= |\xi I - MD(T^{101}(G))| \\ &= \begin{vmatrix} (\xi+n+m-1)I_n - (n+m-1)J_n & -min\{n+m-1, n\}J_{n\times m} \\ -min\{n+m-1, n\}J_{m\times n} & (\xi+n)I_m - nJ_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 38. *Let G be an r-regular graph of order n and size m. Then the degree exponent polynomial of $T^{111}(G)$ is*

$$P_{MD(T^{111}(G))}(\xi) = [\xi - (n+m-1)^2][\xi + (n+m-1)]^{n+m-1}.$$

Proof. The generalized xyz-point-line transformation graph $T^{111}(G)$ of a regular graph G of degree r is a regular graph of degree $n+m-1$. Hence the result follows from (1). □

Theorem 39. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{1+1}(G))}(\xi) &= (\xi+n+m-1)^{n-1}(\xi+n+2r-2)^{m-1}\{\xi^2 - [(n-1)(n+m-1)+(m-1)(n+2r-2)]\xi \\ &\quad +(n-1)(m-1)(n+m-1)(n+2r-2) - min\{n+m-1, n+2r-2\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{1+1}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $R_1 = n+m-1$ and the remaining m vertices are with degree $R_2 = n+2r-2$. Hence

$$MD(T^{1+1}(G)) = \begin{bmatrix} R_1(J_n - I_n) & min\{R_1,R_2\}J_{n\times m} \\ min\{R_1,R_2\}J_{m\times n} & R_2(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{1+1}(G))}(\xi) &= |\xi I - MD(T^{1+1}(G))| \\ &= \begin{vmatrix} (\xi+R_1)I_n - R_1 J_n & -min\{R_1,R_2\}J_{n\times m} \\ -min\{R_1,R_2\}J_{m\times n} & (\xi+R_2)I_m - R_2 J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

Theorem 40. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{1-1}(G))}(\xi) &= (\xi+n+m-1)^{n-1}(\xi+n+m+1-2r)^{m-1}\{\xi^2 - [(n-1)(n+m-1) \\ &\quad +(m-1)(n+m+1-2r)]\xi + (n-1)(m-1)(n+m-1)(n+m+1-2r) \\ &\quad - min\{n+m-1, n+m+1-2r\}^2 mn\}. \end{aligned}$$

Proof. The generalized xyz-point-line transformation graph $T^{1-1}(G)$ of a regular graph G of degree r has two types of vertices. The n vertices with degree $R_1 = n+m-1$ and the remaining m vertices are with degree $R_2 = n+m+1-2r$. Hence

$$MD(T^{1-1}(G)) = \begin{bmatrix} R_1(J_n - I_n) & min\{R_1,R_2\}J_{n\times m} \\ min\{R_1,R_2\}J_{m\times n} & R_2(J_m - I_m) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} P_{MD(T^{1-1}(G))}(\xi) &= |\xi I - MD(T^{1-1}(G))| \\ &= \begin{vmatrix} (\xi+R_1)I_n - R_1 J_n & -min\{R_1,R_2\}J_{n\times m} \\ -min\{R_1,R_2\}J_{m\times n} & (\xi+R_2)I_m - R_2 J_m \end{vmatrix}. \end{aligned}$$

Using Lemma 4, we get the required result. □

The proof of the following theorems are analogous to that of the above.

Theorem 41. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{+01}(G))}(\xi) &= (\xi+m+r)^{n-1}(\xi+n)^{m-1}\{\xi^2-[(n-1)(m+r)+(m-1)n]\xi \\ &\quad +(n-1)(m-1)(m+r)n-min\{n,m+r\}^2mn\}. \end{aligned}$$

Theorem 42. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{+11}(G))}(\xi) &= (\xi+m+r)^{n-1}(\xi+m+n-1))^{m-1}\{\xi^2-[(n-1)(m+r)+(m-1)(m+n-1)]\xi \\ &\quad +(n-1)(m-1)(m+r)(m+n-1)-min\{m+r,m+n-1\}^2mn\}. \end{aligned}$$

Theorem 43. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{++1}(G))}(\xi) &= (\xi+m+r)^{n-1}(\xi+n+2r-2)^{m-1}\{\xi^2-[(n-1)(m+r)+(m-1)(n+2r-2)]\xi \\ &\quad +(n-1)(m-1)(m+r)(n+2r-2)-min\{m+r,n+2r-2\}^2mn\}. \end{aligned}$$

Theorem 44. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{+-1}(G))}(\xi) &= (\xi+m+r)^{n-1}(\xi+n+m+1-2r)^{m-1}\{\xi^2-[(n-1)(m+r)+(m-1)(n+m+1-2r)]\xi \\ &\quad +(n-1)(m-1)(m+r)(n+m+1-2r)-min\{m+r,n+m+1-2r\}^2mn\}. \end{aligned}$$

Theorem 45. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{-01}(G))}(\xi) &= (\xi+n+m-1-r)^{n-1}(\xi+n)^{m-1}\{\xi^2-[(n-1)(n+m-1-r)+(m-1)n]\xi \\ &\quad +n(n-1)(m-1)(n+m-1-r)-min\{n,n+m-1-r\}^2mn\}. \end{aligned}$$

Theorem 46. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{-11}(G))}(\xi) &= (\xi+n+m-1-r)^{n-1}(\xi+n+m-1)^{m-1}\{\xi^2-[(n-1)(n+m-1-r) \\ &\quad +(m-1)(n+m-1)]\xi+(n-1)(m-1)(n+m-1-r)(n+m-1)-(n+m-1)^2mn\}. \end{aligned}$$

Theorem 47. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{MD(T^{-+1}(G))}(\xi) &= (\xi+n+m-1-r)^{n-1}(\xi+n+2r-2)^{m-1}\{\xi^2-[(n-1)(n+m-1-r) \\ &\quad +(m-1)(n+2r-2)]\xi+(n-1)(m-1)(n+m-1-r)(n+2r-2) \\ &\quad -min\{n+m-1-r,n+2r-2\}^2mn\}. \end{aligned}$$

The minimum degree polynomial of subdivision graph (T^{00+}) [9], total graph T^{+++} [9], semitotal point graph (T^{+0+}) [18], semitotal line graph (T^{0++}) [19] can be found in [11].

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