



# Computational Aspects of Determinant and Inverse of Tridiagonal Toeplitz Matrix

Yohanes Mario Defianus Beti <sup>a\*</sup>, Sugi Guritman <sup>b</sup> and Jaharuddin <sup>b</sup>

<sup>a</sup>Magister Applied Mathematics, Faculty of Mathematics and Natural Sciences, IPB University, Indonesia.

<sup>b</sup>Division of Pure Mathematics, Department Mathematics, IPB University, Indonesia.

*Authors' contributions*

*This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.*

*Article Information*

DOI: <https://doi.org/10.9734/jamcs/2024/v39i121951>

**Open Peer Review History:**

This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here:

<https://www.sdiarticle5.com/review-history/126625>

*Received: 14/09/2024*

*Accepted: 20/11/2024*

*Published: 03/12/2024*

**Original Research Article**

## Abstract

In this article, the determinant of tridiagonal Toeplitz matrices is determined recursively and explicitly. The method used is descriptive exploratory the journal written by Fitri Aryani. The inverse of tridiagonal Toeplitz matrices is calculated using the adjoint method, but the determinant and adjoint of the matrices are based on the recursive calculation of the determinant. With this approach, the formulas for the determinant and inverse of tridiagonal Toeplitz matrices can be formulated clearly and efficiently. This study demonstrates the effectiveness of the method used in simplifying computations and provides an algorithm for the formulation.

*Keywords: Tridiagonal toeplitz matrix; determinant; inverse; recursive; explicit.*

\*Corresponding author: E-mail: [yohanesmario@apps.ipb.ac.id](mailto:yohanesmario@apps.ipb.ac.id);

**Cite as:** Beti, Yohanes Mario Defianus, Sugi Guritman, and Jaharuddin. 2024. "Computational Aspects of Determinant and Inverse of Tridiagonal Toeplitz Matrix". *Journal of Advances in Mathematics and Computer Science* 39 (12):84-94. <https://doi.org/10.9734/jamcs/2024/v39i121951>.

**2010 Mathematics Subject Classification:** 53C25, 83C05, 57N16.

## 1 Introduction

The tridiagonal Toeplitz matrix often appears in several applications, such as the discretization of differential equations, which are related to the solution of a difference equation, which is needed to determine the solution of a differential equation problem (Noschese et al., 2013). In addition, the tridiagonal Toeplitz matrix can be used in time series analysis and discrete mathematics (Da Fonseca and Petronilho, 2005). Furthermore, the explicit inverse of a tridiagonal Toeplitz matrix can be used to calculate values such as the trace of the matrix, row sums, and their bounds, as well as to apply it to autoregressive AR(1) models with noise. This is relevant for statistical data modeling and other applications in economics, quantum mechanics, and signal processing (Wang and Zhang, 2016). The characteristic of this matrix is that it has nonzero elements on the main diagonal, subdiagonal/bottom diagonal (first diagonal below the main diagonal), and supra-diagonal/top diagonal (first diagonal above the main diagonal). In contrast, the other elements have a value of 0. Based on this characteristic, the determinant formulation recursively and explicitly, and the inverse can be done with efficient proof steps. The following is an explanation of the tridiagonal Toeplitz matrix. Let  $A_n = \text{ToeTrD}[a, b, c, n]$  be a tridiagonal Toeplitz matrix in order  $n \times n$  of the following form.

$$A_n = \begin{pmatrix} b & c & 0 & \cdots & 0 & 0 & 0 \\ a & b & c & \cdots & 0 & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b & c \\ 0 & 0 & 0 & \cdots & 0 & a & b \end{pmatrix} \tag{1.1}$$

with  $a, b, c \neq 0, \in \mathbb{R}$ . (Salkuyeh, 2006) (Zhang, 2005) (Bhimasankaram et al., 2000)

The problem in the tridiagonal Toeplitz matrices study is determining the matrix's determinant and inverse. Researchers have developed research on the determinant and inverse of the tridiagonal Toeplitz matrix. In Bakti Siregar (Siregar et al., 2014), theorems about the determinant of the Toeplitz matrix, cofactors of the Toeplitz matrix, and the inverse of the Toeplitz matrix were derived. Subsequently, Fitri Aryani (Aryani and Corazon, 2016) derived the theorem about the determinant of the tridiagonal Toeplitz matrix, cofactors of the tridiagonal Toeplitz matrix and used the adjoint method to determine the inverse of the tridiagonal Toeplitz matrix. Based on the background above, this article will discuss the determinant of tridiagonal Toeplitz matrices using recursive and explicit methods and the inverse based on a recursive algorithm. Implementing this theorem's development is expected to exhibit more efficient performance.

The determinant and cofactor of the tridiagonal Toeplitz matrix of the previous research are presented in lemma 1 and lemma 2 as follows.

**Lemma 1.** Given  $A_n = \text{ToeTrD}[a, b, c, n]$  a tridiagonal Toeplitz matrix of size  $n \geq 3$  with  $a, b, c \neq 0, \in \mathbb{R}$  the determinant of the matrix  $A_n$  is

$$\begin{aligned} |A_n| = & b^n - (n-1)ab^{n-2}c + \sum_{i=1}^{n-3} ia^2b^{n-2}c^2 - \left( \sum_{i=1}^1 i + \sum_{i=1}^2 i + \dots + \sum_{i=1}^{n-5} i \right) a^3b^{n-6}c^3 \\ & + \left[ \frac{(n-7)}{1!} \sum_{i=1}^1 i + \frac{(n-8)}{1!} \sum_{i=1}^2 i + \frac{(n-9)}{1!} \sum_{i=1}^3 i + \dots + 1 \sum_{i=1}^{n-7} i \right] a^4b^{n-8}c^4 \\ & - \left[ \frac{(n-9)(n-8)}{2!} \sum_{i=1}^1 i + \frac{(n-10)(n-9)}{2!} \sum_{i=1}^2 i + \dots + 1 \sum_{i=1}^{n-9} i \right] a^5b^{n-10}c^5 \\ & + \left[ \frac{(n-11)(n-10)(n-9)}{3!} \sum_{i=1}^1 i + \frac{(n-12)(n-11)(n-10)}{3!} \sum_{i=1}^2 i \right. \\ & \left. + \dots + 1 \sum_{i=1}^{n-11} i \right] a^6b^{n-11}c^6 \\ & - \left[ \frac{(n-13)(n-12)(n-11)(n-10)}{4!} \sum_{i=1}^1 i \right. \\ & \left. + \frac{(n-14)(n-13)(n-12)(n-11)}{4!} \sum_{i=1}^2 i + \dots + \sum_{i=1}^{n-13} i \right] b^{n-14}c^7 + \dots \end{aligned} \tag{1.2}$$

The cofactor matrix can be determined using the equation  $C_{ij} = (-1)^{i+j}M_{ij}$ , such that the general form of the cofactor matrix becomes a tridiagonal Toeplitz matrix of order  $n \times n$ , as presented in the following Lemma 2.

**Lemma 2.** Given  $A_n = ToeTrD[a, b, c, n]$  a tridiagonal Toeplitz matrix of order  $n \geq 3$  with  $a, b, c \neq 0, \in \mathbb{R}$ , the cofactors of the matrix  $A_n$  is

$$C_n = \begin{pmatrix} (-1)^2|A_{n-1}| & (-1)^3a|A_{n-2}| & \dots & (-1)^na^{n-2}|A_1| & (-1)^{n+1}a^{n-1} \\ (-1)^3c|A_{n-1}| & (-1)^4|A_1||A_{n-2}| & \dots & (-1)^{n+1}a^{n-3}|A_1||A_1| & (-1)^{n+2}a^{n-2}|A_1| \\ (-1)^4c^2|A_{n-3}| & (-1)^4c|A_1||A_{n-3}| & \dots & (-1)^{n+3}a^{n-5}|A_1||A_2| & (-1)^{n+3}a^{n-3}|A_3| \\ (-1)^5c^3|A_{n-4}| & (-1)^6c^3|A_2||A_{n-4}| & \dots & (-1)^{n+3}a^{n-5}|A_1||A_2| & (-1)^{n+4}a^{n-4}|A_3| \\ \vdots & \vdots & \dots & \vdots & \vdots \\ (-1)^na^{n-2}|A_1| & (-1)^{n+1}c^{n-3}|A_1||A_1| & \dots & (-1)^{n-2}|A_1||A_3| & (-1)^{n-2}a|A_{n-2}| \\ (-1)^{n+1}c^{n-1} & (-1)^{n+2}c^{n-2}|A_1| & \dots & (-1)^{n-1}c|A_{n-2}| & (-1)^{2n}c|A_{n-1}| \end{pmatrix} \tag{1.3}$$

From the cofactor matrix above, the adjoint of the matrix can be determined as follows.

$$adj(A_n) = \begin{pmatrix} (-1)^2|A_{n-1}| & (-1)^3c|A_{n-2}| & \dots & (-1)^na^{n-2}|A_1| & (-1)^{n+1}c^{n-1} \\ (-1)^3a|A_{n-1}| & (-1)^4|A_1||A_{n-2}| & \dots & (-1)^{n+1}c^{n-3}|A_1||A_1| & (-1)^{n+2}c^{n-2}|A_1| \\ (-1)^4a^2|A_{n-3}| & (-1)^4a|A_1||A_{n-3}| & \dots & (-1)^{n+3}c^{n-5}|A_1||A_2| & (-1)^{n+3}c^{n-3}|A_3| \\ (-1)^5a^3|A_{n-4}| & (-1)^6a^3|A_2||A_{n-4}| & \dots & (-1)^{n+3}c^{n-5}|A_1||A_2| & (-1)^{n+4}c^{n-4}|A_3| \\ \vdots & \vdots & \dots & \vdots & \vdots \\ (-1)^na^{n-2}|A_1| & (-1)^{n+1}a^{n-3}|A_1||A_1| & \dots & (-1)^{n-2}|A_1||A_3| & (-1)^{n-2}c|A_{n-2}| \\ (-1)^{n+1}a^{n-1} & (-1)^{n+2}a^{n-2}|A_1| & \dots & (-1)^{n-1}a|A_{n-2}| & (-1)^{2n}c|A_{n-1}| \end{pmatrix} \tag{1.4}$$

After determining the determinant and adjoint of the tridiagonal Toeplitz matrix, the inverse of the tridiagonal Toeplitz matrix can be determined as follows:

$$A_n^{-1} = \frac{1}{det(A)}adj(A) \tag{1.5}$$

(Heinig and Rost, 1984) (Gray, 2005)

Based on the explanation of the above theorem, determining the determinant is explicitly defined as the value of a single matrix determinant. Meanwhile, the determinant is explicitly determined when finding the cofactor matrix to determine the inverse repeatedly. Therefore, in this article, a theorem related to the recursive determination of the determinant will be constructed, which involves determining subsets of determinants of matrices. This data is then utilized to determine the inverse of tridiagonal Toeplitz matrices.

## 2 The Determinant and Inverse Formulation

In this section, we will discuss determinants, inverses, and algorithms for determining the determinant and inverse of tridiagonal Toeplitz matrices.

### 2.1 The determinant of The Tridiagonal Toeplitz Matrix

The following will be derived from the formula for calculating the determinant of a tridiagonal Toeplitz matrix, both recursively and explicitly.

**Theorem 1.** The determinant of the tridiagonal Toeplitz matrix  $A_n = ToeTrD[a, b, c, n]$ , denoted as  $det(A_n) = d_n$ , and  $d_n$  is obtained recursively as follows:

Recursive basis:  $d_1 = b$  and  $d_2 = b^2 - x$ , with  $x = ac$ .

Recursive process: For all  $n \in \mathbb{Z}$  and  $n \geq 3$ ,  $d_n = bd_{n-1} - xd_{n-2}$ .

*Proof.* Proof of the theorem using direct proof.

Recursive basis:  $d_1 = b$  and  $d_2 = b^2 - x$ , with  $x = ac$ .

Recursive process for  $n = 3$ :

$$\begin{aligned} d_3 &= \begin{vmatrix} b & c & 0 \\ a & b & c \\ 0 & a & b \end{vmatrix} \\ &= b \begin{vmatrix} b & c \\ a & b \end{vmatrix} - c \begin{vmatrix} a & c \\ 0 & b \end{vmatrix} \\ &= bd_2 - c(ab) \\ &= bd_2 - xd_1, \quad \text{with } x = ac \end{aligned}$$

In general, for  $n \in \mathbb{N}$ , such that

$$\begin{aligned} d_n &= b \begin{vmatrix} b & c & 0 & \cdots & 0 & 0 \\ a & b & c & \cdots & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a & b & c \\ 0 & \cdots & 0 & 0 & a & b \end{vmatrix} - c \begin{vmatrix} a & c & 0 & \cdots & 0 & 0 \\ 0 & b & c & \cdots & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a & b & c \\ 0 & \cdots & 0 & 0 & a & b \end{vmatrix} \\ &= bd_{n-1} - ac(d_{n-2}) \\ &= bd_{n-1} - xd_{n-2}, \quad \text{with } x = ac. \end{aligned}$$

□

**Theorem 2.** The determinant of the tridiagonal Toeplitz matrix  $A_n = \text{ToeTrD}[a, b, c, n]$ , denoted as  $\det(A_n) = d_n$  and  $d_n$  is obtained explicitly as follows:

$$d_n = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{i} b^{n-2i} x^i, \quad \forall n \in \mathbb{N}$$

*Proof.* Proof of the theorem by using mathematical induction(Burden and Faires, 2020).

Basic Induction :  $n = 1$

$$\begin{aligned} \det(A_1) &= |A_1| = |b| = b \\ &= (-1)^0 \binom{2-1}{1} b^1 x^0 \end{aligned}$$

Induction step:

Assume  $d_k = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k-i}{i} b^{k-2i} x^i$ , true  $\forall k \in \mathbb{N}$ , will be proven correct for

$$d_{k+1} = \sum_{i=0}^{\lfloor (k+1)/2 \rfloor} (-1)^i \binom{k+1-i}{i} b^{k+1-2i} x^i.$$

In this section, two cases will be proven.

1. For the case  $k$  is even, let  $\lfloor \frac{k}{2} \rfloor = \lfloor \frac{k+1}{2} \rfloor = \frac{k}{2}$  and  $\lfloor \frac{k-1}{2} \rfloor = \frac{k}{2} - 1$ , then will be proven correct for  $d_{k+1}$ 

$$= \sum_{i=0}^{\lfloor (k+1)/2 \rfloor} (-1)^i \binom{k+1-i}{i} b^{k+1-2i} x^i.$$

Based on Theorem 1,  $d_{k+1} = bd_k - xd_{k-1}$ , we have

$$\begin{aligned} d_{k+1} &= b \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k-i}{i} b^{k-2i} x^i - x \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} (-1)^i \binom{k-1-i}{i} b^{k-1-2i} x^i \\ &= \sum_{i=0}^{k/2} (-1)^i \binom{k-i}{i} b^{k+1-2i} x^i - \sum_{i=0}^{k/2-1} (-1)^i \binom{k-1-i}{i} b^{k-1-2i} x^{i+1} \\ &= b^{k+1} - \left[ \binom{k-1}{1} + \binom{k-1}{0} \right] b^{k-1} x + \left[ \binom{k-2}{2} + \binom{k-2}{1} \right] b^{k-3} x^2 \\ &\quad - \dots + (-1)^{k/2} \left[ \binom{k/2}{k/2} + \binom{k/2}{k/2-1} \right] b^k x^{k/2} \end{aligned}$$

if and only if

$$\begin{aligned} d_{k+1} &= b^{k+1} + \sum_{i=1}^{k/2} (-1)^i \left[ \binom{k-i}{i} + \binom{k-i}{i-1} \right] b^{k+1-2i} x^i \\ &= \sum_{i=1}^{\lfloor k+1/2 \rfloor} (-1)^i \binom{k+1-i}{i} b^{k+1-2i} x^i \end{aligned}$$

2. For the case of  $k$  is odd, let  $\lfloor \frac{k+1}{2} \rfloor = \frac{k+1}{2}$  and  $\lfloor \frac{k}{2} \rfloor = \lfloor \frac{k-1}{2} \rfloor = \frac{k-1}{2}$ , will be proven correct for  $d_{k+1}$
- $$= \sum_{i=0}^{\lfloor k+1/2 \rfloor} (-1)^i \binom{k+1-i}{i} b^{k+1-2i} x^i.$$

Based on Theorem 1,  $d_{k+1} = bd_k - xd_{k-1}$ , we have

$$\begin{aligned} d_{k+1} &= b \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k-i}{i} b^{k-2i} x^i - x \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} (-1)^i \binom{k-1-i}{i} b^{k-1-2i} x^i \\ &= b^{k+1} - \binom{k-1}{1} b^{k-1} x + \binom{k-2}{2} b^{k-3} x^2 - \binom{k-3}{3} b^{k-5} x^3 + \dots \\ &\quad + (-1)^{(k-1)/2} b^2 x^{(k-1)/2} \\ &\quad - \left( b^{k-1} x + \binom{k-2}{1} b^{k-3} x^2 - \binom{k-3}{2} b^{k-5} x^3 + \dots + (-1)^{(k-1)/2} x^{(k+1)/2} \right) \\ &= b^{k+1} - \left[ \binom{k-1}{1} + \binom{k-1}{0} \right] b^{k-1} x + \left[ \binom{k-2}{2} + \binom{k-2}{1} \right] b^{k-3} x^2 \\ &\quad - \dots + (-1)^{(k-1)/2} \left[ \binom{k-(k-1)/2}{(k-1)/2} + \binom{k-(k-1)/2}{k-3/2} \right] b^2 x^{(k-1)/2} \end{aligned}$$

if and only if

$$\begin{aligned} d_{k+1} &= b^{k+1} + \sum_{i=1}^{k+1/2} (-1)^i \left[ \binom{k-i}{i} + \binom{k-i}{i-1} \right] b^{k+1-2i} x^i \\ &= \sum_{i=1}^{\lfloor k+1/2 \rfloor} (-1)^i \binom{k+1-i}{i} b^{k+1-2i} x^i \end{aligned}$$

so,  $d_n = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{i} b^{n-2i} x^i$ ,  $\forall n \in \mathbb{N}$ . □

## 2.2 The inverse of The Tridiagonal Toeplitz Matrix

An exploratory illustration will be provided for cases  $n = 6$  and  $n = 7$  to determine the inverse of a tridiagonal Toeplitz matrix.

**Lemma 3.** Let a tridiagonal Toeplitz matrix  $A_6 = \text{ToeTrD}[a, b, c, 6]$ .

If  $[d_1, d_2, d_3, d_4, d_5, d_6]$  sequence of submatrix determinants  $A_6$ , which has been computed based on Theorem 1, the inverse of matrix  $A_6$  can also be efficiently formulated as

$$A^{-1} = \frac{1}{d_6} \begin{pmatrix} d_5 & -cd_4 & c^2d_3 & -c^3d_2 & c^4d_1 & -c^5 \\ -ad_4 & d_1d_4 & -cd_1d_3 & c^2d_1d_2 & -c^3d_1^2 & c^4d_1 \\ a^2d_3 & -ad_1d_4 & d_2d_3 & -cd_2^2 & c^2d_1d_2 & -c^3d_2 \\ -a^3d_1 & a^2d_1d_2 & -ad_2^2 & d_2d_3 & -cd_1d_3 & c^2d_3 \\ a^4d_1 & -a^3d_1^2 & a^2d_1d_2 & -ad_1d_3 & d_1d_4 & -cd_4 \\ -a^5 & a^4d_1 & -a^3d_2 & a^2d_3 & -ad_4 & d_5 \end{pmatrix}$$

*Proof.* According to the definition of the inverse,  $A^{-1} = \frac{1}{d_6} (\alpha_{i,j})_{i,j=6}^6$  with  $\alpha_{i,j} = (-1)^{i+j} M_{ij}$  and  $M_{ij}$  is minor of row  $i$  to  $-j$ , column  $i$  to  $-j$  from matrix  $A$  for all  $i, j = 1, 2, 3, \dots, 6$  applies :

1. The entries of the main diagonal  $\alpha_{i,i} = \alpha_{7-i,7-i}$  for  $i = 1, 2, 3$  and formulated as follows.  
 $\alpha_{1,1} = \alpha_{6,6} = d_5 \quad \alpha_{2,2} = \alpha_{5,5} = d_1d_4 \quad \alpha_{3,3} = \alpha_{4,4} = d_2d_3$
2. In the entries of  $i + j = 3$  and  $i \neq j$ , obtained  $\alpha_{i,j} = \alpha_{7-j,7-i}$  and formulated as follows.  
 $\alpha_{1,2} = \alpha_{5,6} = -cd_4 \quad \alpha_{2,1} = \alpha_{6,5} = -ad_4$
3. In the entries of  $i + j = 4$  and  $i \neq j$ , obtained  $\alpha_{i,j} = \alpha_{7-j,7-i}$  and formulated as follows.  
 $\alpha_{1,3} = \alpha_{4,6} = c^2d_3 \quad \alpha_{3,1} = \alpha_{6,4} = a^2d_3$
4. In the entries of  $i + j = 5$  and  $i \neq j$ , obtained  $\alpha_{i,j} = \alpha_{7-j,7-i}$  and formulated as follows.  
 $\alpha_{1,4} = \alpha_{3,6} = -c^3d_2 \quad \alpha_{4,1} = \alpha_{6,3} = -a^3d_2$   
 $\alpha_{2,3} = \alpha_{4,5} = -cd_1d_3 \quad \alpha_{3,2} = \alpha_{5,4} = -ad_1d_3$
5. In the entries of  $i + j = 6$  and  $i \neq j$ , obtained  $\alpha_{i,j} = \alpha_{7-j,7-i}$  and formulated as follows.  
 $\alpha_{1,5} = \alpha_{2,6} = c^4d_1 \quad \alpha_{5,1} = \alpha_{6,2} = a^4d_1$   
 $\alpha_{2,4} = \alpha_{3,5} = c^2d_1d_2 \quad \alpha_{4,2} = \alpha_{5,3} = a^2d_1d_2$
6. In the entries of  $i + j = 7$  and  $i \neq j$ , obtained  $\alpha_{i,j} = \alpha_{7-j,7-i}$  and formulated as follows.  
 $\alpha_{1,6} = -c^5 \quad \alpha_{6,1} = -a^5$   
 $\alpha_{2,5} = -c^3d_1^2 \quad \alpha_{5,2} = -a^3d_1^2$   
 $\alpha_{3,4} = -cd_2^2 \quad \alpha_{4,3} = -ad_2^2$

□

**Lemma 4.** Let a tridiagonal Toeplitz matrix  $A_7 = \text{ToeTrD}[a, b, c, 7]$ .

If  $[d_1, d_2, d_3, d_4, d_5, d_6, d_7]$  sequence of submatrix determinants  $A_7$ , which has been computed based on Theorem 1, the inverse of matrix  $A_7$  can also be efficiently formulated as

$$A_7^{-1} = \frac{1}{d_7} \begin{pmatrix} d_6 & -cd_5 & c^2d_4 & -c^3d_3 & c^4d_2 & -c^5d_1 & c^6 \\ -ad_5 & d_1d_5 & -cd_1d_4 & c^2d_1d_3 & -c^3d_1d_2 & c^4d_1^2 & -c^5d_1 \\ a^2d_4 & -ad_1d_4 & d_2d_4 & -cd_2d_3 & c^2d_2^2 & -c^3d_1d_2 & c^4d_2 \\ -a^3d_3 & a^2d_1d_3 & -ad_2d_3 & d_3^2 & -cd_2d_3 & c^2d_1d_3 & -c^3d_3 \\ a^4d_2 & -a^3d_1d_2 & a^2d_2^2 & -ad_2d_3 & d_2d_4 & -cd_1d_4 & c^2d_4 \\ -a^5d_1 & a^4d_1^2 & -a^3d_1d_2 & a^2d_1d_3 & ad_1d_4 & d_1d_5 & cd_5 \\ a^6 & -a^5d_1 & a^4d_2 & -a^3d_3 & a^2d_4 & -ad_5 & d_6 \end{pmatrix}$$

*Proof.* According to the definition of the inverse, it can be written that

$A^{-1} = \frac{1}{d_7} (\alpha_{i,j})_{i,j=7}^7$  with  $\alpha_{i,j} = (-1)^{i+j} M_{ij}$  and  $M_{ij}$  is the minor of row to  $-j$ , column to  $-i$  from matrix  $A$  for all  $i, j = 1, 2, 3, \dots, 7$  we have :

1. In the entries of main diagonal  $\alpha_{i,i} = \alpha_{8-i,8-i}$  for  $i = 1, 2, 3, 4$  and formulated as follows.
 
$$\alpha_{1,1} = \alpha_{7,7} = d_6 \quad \alpha_{2,2} = \alpha_{6,6} = d_1 d_5$$

$$\alpha_{3,3} = \alpha_{5,5} = d_2 d_4 \quad \alpha_{4,4} = d_3^2$$
2. In the entries of  $i + j = 3$  and  $i \neq j$ , obtained  $\alpha_{i,j} = \alpha_{8-j,8-i}$  and formulated as follows.
 
$$\alpha_{1,2} = \alpha_{6,7} = -cd_5 \quad \alpha_{2,1} = \alpha_{7,6} = -ad_5$$
3. In the entries of  $i + j = 4$  and  $i \neq j$ , obtained  $\alpha_{i,j} = \alpha_{8-j,8-i}$  and formulated as follows.
 
$$\alpha_{1,3} = \alpha_{5,7} = c^2 d_4 \quad \alpha_{3,1} = \alpha_{7,5} = a^2 d_4$$
4. In the entries of  $i + j = 5$  and  $i \neq j$ , obtained  $\alpha_{i,j} = \alpha_{8-j,8-i}$  and formulated as follows.
 
$$\alpha_{1,4} = \alpha_{4,7} = -c^3 d_3 \quad \alpha_{4,1} = \alpha_{7,4} = -a^3 d_3$$

$$\alpha_{2,3} = \alpha_{5,6} = -cd_1 d_4 \quad \alpha_{3,2} = \alpha_{6,5} = -ad_1 d_4$$
5. In the entries of  $i + j = 6$  and  $i \neq j$ , obtained  $\alpha_{i,j} = \alpha_{8-j,8-i}$  and formulated as follows.
 
$$\alpha_{1,5} = \alpha_{3,7} = c^4 d_2 \quad \alpha_{5,1} = \alpha_{7,3} = a^4 d_2$$

$$\alpha_{2,4} = \alpha_{4,6} = c^2 d_1 d_3 \quad \alpha_{4,2} = \alpha_{6,4} = a^2 d_1 d_3$$
6. In the entries of  $i + j = 7$  and  $i \neq j$ , obtained  $\alpha_{i,j} = \alpha_{8-j,8-i}$  and formulated as follows.
 
$$\alpha_{6,1} = \alpha_{2,7} = -c^5 d_1 \quad \alpha_{1,6} = \alpha_{7,2} = -a^5 d_1$$

$$\alpha_{5,2} = \alpha_{3,6} = -c^3 d_1 d_2 \quad \alpha_{5,2} = \alpha_{6,3} = -a^3 d_1 d_2$$

$$\alpha_{3,4} = \alpha_{4,5} = -cd_2 d_3 \quad \alpha_{4,3} = \alpha_{5,4} = -ad_2 d_3$$
7. In the entries of  $i + j = 8$  and  $i \neq j$ , obtained  $\alpha_{i,j} = \alpha_{8-j,8-i}$  and formulated as follows.
 
$$\alpha_{1,7} = c^6 \quad \alpha_{7,1} = a^6$$

$$\alpha_{2,6} = c^4 d_1^2 \quad \alpha_{6,2} = a^4 d_1^2$$

$$\alpha_{3,5} = c^2 d_2^2 \quad \alpha_{5,3} = a^2 d_2^2$$

□

From the results of the explorations in Lemma 3 and Lemma 4, the inverse of the tridiagonal Toeplitz matrix  $A_n$  for each  $n \in \mathbb{N}$  is provided in the following theorem.

**Theorem 3.** Let a tridiagonal Toeplitz matrix  $A_n = \text{ToeTrD}[a, b, c, n]$ .

If  $[d_1, d_2, \dots, d_{n-1}, d_n]$  sequence of submatrix determinants  $A_n$ , which has been computed based on Theorem 1, and defined  $d_0 = 1$ , then the inverse of matrix  $A_n$  is

$$A^{-1} = \frac{1}{d_n} (\alpha_{i,j})_{i,j=n}^n$$

with:

1. In entries of the main diagonal  $\alpha_{i,j} = \alpha_{n+1-i,n+1-i} = d_{i-1} d_{n-i}$  with  $i = 1, 2, \dots, \lfloor \frac{n+1}{2} \rfloor$
2. The entries matrix is formulated as follows:

(a)  $\alpha_{i,j} = \alpha_{n+1-j,n+1-i} = kc^{j-i}$

(b)  $\alpha_{j,i} = \alpha_{n+1-i,n+1-j} = ka^{j-i}$

for  $i = 1, 2, \dots, \lfloor \frac{n+1}{2} \rfloor$  and  $j = (i + 1), (i + 2), \dots, (n - i + 1)$  with  $k = (-1)^{i+j} d_{i-1} d_{n-i+j}$

*Proof.* Inverse of the tridiagonal Toeplitz matrix  $A_n$  can be expressed as

$A^{-1} = \frac{1}{d_n} (\alpha_{i,j})_{i,j=n}^n$  with  $\alpha_{i,j} = (-1)^{j+i} M_{ji}$  and  $M_{ji}$  is the minor of row to  $-j$ , column to  $-i$  from matrix  $A$  for all  $i, j = 1, 2, \dots, n$  subsequently applies :

1. In the main diagonal entries can be expressed as  $\alpha_{ii} = \alpha_{n+1-i, n+1-i}$  for  $i = 1, 2, \dots, \lfloor \frac{n+1}{2} \rfloor$  and formulated as follows:

$$\alpha_{1,1} = \alpha_{n,n} = d_0 d_{n-1} = d_{n-1}$$

⋮

$$\alpha_{\lfloor \frac{n+1}{2} \rfloor, \lfloor \frac{n+1}{2} \rfloor} = \alpha_{n-1, n-1} = \alpha_{n+1-\lfloor \frac{n+1}{2} \rfloor, n+1-\lfloor \frac{n+1}{2} \rfloor} = d_{\lfloor \frac{n+1}{2} \rfloor-1} d_{n-\lfloor \frac{n+1}{2} \rfloor}$$

2. In entries of the matrix for  $i = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$  and  $j = (i+1), (i+2), \dots, (n-i+1)$  can be formulated as follows.

(a)  $\alpha_{i,j} = \alpha_{n+1-j, n+1-i}$ , with  $k = (-1)^{i+j} d_{i-1} d_{n-1-j+i}$

to prove this case:

i. for  $i = 1$  and  $j = (i+1), (i+2), \dots, (n-i+1)$ , so  $\alpha_{i,j}$  can be formulated as follows.

$$\alpha_{1,2} = \alpha_{n-1, n} = -kc^{2-1} = -cd_0 d_{n-2} = -cd_{n-2}$$

⋮

$$\alpha_{1, n} = (-1)^{n-1} kc^{n-1} = (-1)^{n-1} c^{n-1} d_0 d_0 = (-1)^{n-1} c^{n-1}$$

ii. In entries of matrix for  $i = 2$  and  $j = (i+1), (i+2), \dots, (n-i+1)$ ,  $\alpha_{i,j}$  can be formulated as follows.

$$\alpha_{2,3} = \alpha_{n-2, n-1} = -kc^{3-2} = -cd_1 d_{n-2}$$

⋮

$$\alpha_{2, n-1} = (-1)^{n+1} kc^{n-3} = (-1)^{n+1} c^{n-3} d_1 d_2$$

⋮

In entries of matrix for  $i = \lfloor \frac{n}{2} \rfloor$  and  $j = (i+1), (i+2), \dots, (n-i+1)$ ,  $\alpha_{i,j}$  can be formulated as follows.

$$\alpha_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1} = \alpha_{n+\lfloor \frac{n}{2} \rfloor, n+1-\lfloor \frac{n}{2} \rfloor} = (-1)^{2\lfloor \frac{n}{2} \rfloor+1} kc = (-1)^{2\lfloor \frac{n}{2} \rfloor+1} cd_{\lfloor \frac{n}{2} \rfloor-1} d_{n-2}$$

⋮

$$\alpha_{\lfloor \frac{n}{2} \rfloor, n-\lfloor \frac{n}{2} \rfloor + 1} = (-1)^{n+1} kc^{n-2\lfloor \frac{n}{2} \rfloor+1} = (-1)^{n+1} c^{n-2\lfloor \frac{n}{2} \rfloor+1} d_{\lfloor \frac{n}{2} \rfloor-1} d_{2\lfloor \frac{n}{2} \rfloor+2}$$

(b)  $\alpha_{j,i} = \alpha_{n+1-i, n+1-j}$  with  $k = (-1)^{i+j} d_{i-1} d_{n-1-j+i}$

to prove this case:

i. In entries of matrix for  $i = 1$  and  $j = (i+1), (i+2), \dots, (n-i+1)$ ,  $\alpha_{i,j}$  can be formulated as follows.

$$\alpha_{2,1} = \alpha_{n, n-1} = -ka^{2-1} = -cd_0 d_{n-2} = -ad_{n-2}$$

⋮

$$\alpha_{n,1} = (-1)^{n+1} ka^{n-1} = (-1)^{n+1} a^{n-1} d_0 d_0 = (-1)^{n+1} a^{n-1}$$

ii. In entries of matrix for  $i = 2$  and  $j = (i+1), (i+2), \dots, (n-i+1)$ ,  $\alpha_{i,j}$  can be formulated as follows.

$$\alpha_{3,2} = \alpha_{n-1, n-2} = -ka^{3-2} = -ad_1 d_{n-2} = -ad_1 d_{n-2}$$

⋮

$$\alpha_{n-1,2} = (-1)^{n+1} ka^{n-3} = (-1)^{n+1} a^{n-3} d_1 d_2$$

⋮

In entries of matrix for  $i = \lfloor \frac{n}{2} \rfloor$  and  $j = (i+1), (i+2), \dots, (n-i+1)$ ,  $\alpha_{i,j}$  can be formulated as follows.

$$\alpha_{\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor} = \alpha_{n+1-\lfloor \frac{n}{2} \rfloor, n-\lfloor \frac{n}{2} \rfloor} = (-1)^{2\lfloor \frac{n}{2} \rfloor+1} ka = (-1)^{2\lfloor \frac{n}{2} \rfloor+1} ad_{\lfloor \frac{n}{2} \rfloor-1} d_{n-2}$$

⋮

$$\alpha_{n-\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor} = (-1)^{n+1} ka^{n-2\lfloor \frac{n}{2} \rfloor+1} = (-1)^{n+1} a^{n-2\lfloor \frac{n}{2} \rfloor+1} d_{\lfloor \frac{n}{2} \rfloor-1} d_{2\lfloor \frac{n}{2} \rfloor+2}$$

□



### 2.3 Computational remark

In this subsection, we provide a simple illustration to explain the formula for calculating the determinant based on Theorem 1 and the inverse based on Theorem 3. Then, considering that illustration, we construct an algorithm.

**Example 1.** (Simple illustration) Let  $A_5 = [1, -2, 1, 5]$  with  $b = -2, a = c = 1$ . Using Theorem 1, we obtain the  $[-2, 3, -4, 5, -6]$ , where  $d_0 = 1$  is define. Applying Theorem 3, the inverse of the matrix  $A_5$  is given by:

$$A^{-1} = -\frac{1}{6} \begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

From the illustration above, the determinant of matrix is determined recursively, resulting in a subset of the matrix determinant. To determine the determinant, some data can be stored for the subsequent process of calculating the inverse. Therefore, the computational process can be performed within single function and in parallel to achieve a high-speed and efficient performance.

**Algorithm 1.** INPUT 1 :  $DetToeTrD[a,b,c,n]$

OUTPUT 1 :  $[d_1, d_2, d_3, \dots, d_n]$ ; subdeterminan of tridiagonal Toeplitz matrix

1.  $x \leftarrow a \cdot c$ ;  $d_1 \leftarrow b$ ;  $d_2 \leftarrow b^2 - x$ .

2.  $L \leftarrow [d_1, d_2]$ ;  $j \leftarrow 2$

3. for  $i$  while  $j < n$  do

$u \leftarrow b \cdot L[j] - x \cdot L[j - 1]$

$L \leftarrow [op(L), u]$

$j \leftarrow j + 1$

end do

4. return(L)

(Golub and Loan, 2013) (Bini and Pan, 1994)

The data generated by Algorithm 1 will be utilized to determine the inverse in the following Algorithm 2.

**Algorithm 2.** INPUT 2 :  $InvMToeTrD[a,b,c,n]$

OUTPUT 2 : inverse tridiagonal Toeplitz matrix  $A_n^{-1}$

1.  $SeqDet \leftarrow DetMToeTrD(a,b,c,n)$ ;  $dt \leftarrow SeqDet[n]$

2.  $L \leftarrow [1, op(1..n-1, SeqDet)]$ ;  $C \leftarrow [seq(c^i, i = 0..n-1)]$

3.  $A \leftarrow [seq(a^i, i = 0..n-1)]$

4.  $M \leftarrow Matrix(n)$

5.  $m \leftarrow floor(n/2)$

6. for  $i$  from 1 to  $m$  do

$u \leftarrow \frac{L[i] \cdot L[n+1-i]}{dt}$

$M[i, i] \leftarrow u$ ;  $M[n+1-i, n+1-i] \leftarrow u$

for  $j$  from  $i+1$  to  $(n-i)$  do

$k \leftarrow (-1)^{i+j} \cdot L[i] \cdot L[n-j+1]$

```

    u ← k · C[j - i + 1]/dt
    M[i, j] ← u    M[n + 1 - j, n + 1 - i] ← u
    u ← k · A[j - i + 1]/dt
    M[j, i] ← u;    M[n + 1 - i, n + 1 - j] ← u
end do
l ← n - i + 1
k ← (-1)i+1 · L[i] · L[n - l + 1]
u ← k · C[l - i + 1]/dt;    M[i, l] ← u
u ← k · A[l - i + 1]/dt;    M[l, i] ← u
end do
7. if n mod 2 = 1 then
    i ← m + 1
    u ← (L[i] · L[i])/dt
    M[i, i] ← u
end if
8. return(M)
(Bini and Pan, 1994)

```

### 3 Concluding Remark

The determinant of the tridiagonal Toeplitz matrix can be determined recursively and explicitly for a given size. Both types of determinants are presented in Theorem 1 and Theorem 2. Theorem 2 represents a simplified form of Lemma 1 as written by Aryani and Corazon.

In determining the inverse of the tridiagonal Toeplitz matrix, based on the recursive determinant is also presented in Theorem 3. This subset of determinants is used to determine the entries of the inverse matrix based on established rules. Finally, we develop an algorithm to find inverse of the tridiagonal Toeplitz matrix utilizing the recursive method in determinant computation.

#### Disclaimer (Artificial Intelligence)

Author(s) hereby declare that generative AI technologies such as Large Language Models, etc have been used during writing or editing of manuscripts. This explanation will include the name, version, model, and source of the generative AI technology and as well as all input prompts provided to the generative AI technology.

Details of the AI usage are given below:

1. Name and Version: ChatGPT
2. Usage: Used to check and refine the syntax of Python code for calculating the determinant and inverse of tridiagonal Toeplitz matrix.
3. Input Prompts: Check the syntax and correctness of this Python code for finding the determinant of a tridiagonal Toeplitz matrix. [The code can be seen on algorithms 1 and 2]

#### Competing Interests

Authors have declared that no competing interests exist.

## References

- Aryani, F. and Corazon, C. M. (2016). Inverse of tridiagonal toeplitz matrix by adjoin method. In *Proceeding Ico-stechs*.
- Bhimasankaram, P., Murthy, K. R., and Sinha, A. (2000). On the inverses of certain tridiagonal toeplitz matrices. *Linear Algebra and its Applications*, 326(1-3):153–160.
- Bini, D. A. and Pan, V. Y. (1994). *Polynomial and matrix computations. Volume 1: Fundamental Algorithms*. Springer-Verlag, New York.
- Burden, R. L. and Faires, J. D. (2020). Numerical analysis.
- Da Fonseca, C. M. and Petronilho, J. (2005). Explicit inverse of a tridiagonal k-toeplitz matrix. *Numerische Mathematik*, 100(3):457–482.
- Golub, G. H. and Loan, C. F. V. (2013). *Matrix Computations*. Johns Hopkins University Press, Baltimore, MD, 4th edition.
- Gray, R. M. (2005). *Toeplitz and Circulant Matrices*. Department of Electrical Engineering, Stanford University.
- Heinig, G. and Rost, K. (1984). Algebraic methods for toeplitz-like matrices and operators.
- Noschese, S., Pasquini, L., and Reichel, L. (2013). Tridiagonal toeplitz matrices: Properties and novel applications. *Numerical Linear Algebra with Applications*, 20(2):306–326.
- Salkuyeh, D. K. (2006). Positive integer power of the tridiagonal matrices toeplitz. *International Mathematical Forum*, 1(22):1061–1065.
- Siregar, B., Tulus, T., and Nasution, W. (2014). Inverse of a toeplitz matrix using the adjoin method. *Saintia Matematika*, 2(1):85–94.
- Wang, Z. and Zhang, W. (2016). Explicit inverse of a special tridiagonal matrix and its application. *Applied Mathematics and Computation*, 274:81–87.
- Zhang, F. (2005). The schur complement and its applications. *Numerical Mathematics and Computing*, 4(1):27–39.

**Disclaimer/Publisher's Note:** The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of the publisher and/or the editor(s). This publisher and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.

© Copyright (2024): Author(s). The licensee is the journal publisher. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

**Peer-review history:**

The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)  
<https://www.sdiarticle5.com/review-history/126625>