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On the Extended (k, t)–Fibonacci Numbers

Sergio Falcon [a*]

[a] *Department of Mathematics, Universidad de as Palmas de Gran Canaria, Spain.*

Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

This article studies an extension of the concept of k–Fibonacci numbers by introducing a new non-zero positive integer parameter t. In case $t = 1$, the numbers found are the Leonardo numbers. A homogeneous recurrence relationship is found between these new numbers, and various formulas are studied such as the Binet Identity or the generating function.

Keywords: k–Fibonacci numbers; Binet identity; Recurrence relation; Generating function; Leonardo numbers.

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**Corresponding author: E-mail: sergio.falcon@ulpgc.es;*

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1 Introduction

One of the more studied sequences is the Fibonacci sequence [1], and it has been generalized in many ways [2, 3, 4, 5]. Here, we use the following one-parameter generalization of the Fibonacci sequence.

Definition 1.1. For any integer number $k \geq 1$, the k–Fibonacci number is defined as $F_{k,n} = k\,F_{k,n-1} + F_{k,n-2}$ with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$.

Thes numbers generates the integer sequence $\{F_{k,n}\} = \{0, 1, k, k^2+1, k^3+2k, k^4+3k^2+1, \ldots\}$ Note that for $k = 1$ the classical Fibonacci sequence is obtained $\{F_n\} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$ indexed in the OEIS [6] as A000045.

For $k = 2$ we obtain the Pell sequence $\{P_n\} = \{0, 1, 2, 5, 12, 29, 70, 179, 408, \ldots\}$, A000129 in the OEIS. Some of the properties that the k–Fibonacci numbers verify and that we will need later are summarized bellow [7, 8, 9]. In particular, and since we will use it throughout this article, we indicate the Binet identity,

$$F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} \tag{1.1}$$

where $\sigma_1 = \dfrac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2 = \dfrac{k-\sqrt{k^2+4}}{2}$ are the characteristic roots of the relation of the definition. For the classical Fibonacci numbers ($k = 1$), it is $\sigma_1 = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio ϕ while , $\sigma_2 = \dfrac{1-\sqrt{5}}{2} = \psi$
Among other properties, these roots verify $\sigma_1 + \sigma_2 = k$, $\sigma_1 \cdot \sigma_2 = -1$, $\sigma^2 = k\,\sigma + 1$, $\sigma_1 - \sigma_2 = \sqrt{k^2+4}$
The sum of the first n numbers is given by the formula $\displaystyle\sum_{j=0}^{n} F_{k,j} = \frac{F_{k,n} + F_{k,n+1} - 1}{k}$ that, for the classical Fibonacci sequence ($k = 1$) is $\displaystyle\sum_{j=0}^{n} F_j = F_{n+2} - 1$
The generating function of the k–Fibonacci numbers is $f(k, x) = \dfrac{x}{1 - k\,x - x^2}$ and the negative k–Fibonacci numbers are $F_{k,-n} = (-1)^{n+1} F_{k,n}$.

1.1 Leonardo numbers

A leonardo number is defined bay mean of the recurrence relation $L_n = L_{n-1} + L_{n-2} + 1$ with initial conditions $L_0 = 1$ and $L_1 = 1$. The sequence of Leonardo numbers is $\{L_n\} = \{1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, \ldots\}$, A001595 in the OEIS. The Leonardo numbers are related to the Fibonacci numbers as $L_n = 2F_{n+1} - 1$

2 On the Extended (k, t)–Fibonacci Numbers

In this section we generalize the recurrence relation that the k–Fibonavcci numbers must satisfy by adding a complementary term that is a positive integer constant t. Later, we will relate these numbers to the Leonardo numbers.

Definition 2.1. Let t be a positive integer number, $t \in N$. It defines the linear non–homogeneus recurrence relation as

$$T(k, t, n) = k\,T(k, t, n-1) + T(k, t, n-2) + t \tag{2.1}$$

It is necessary to indicate two initial conditions in order to determine exactly the terms of this sequence. According to this definition, this sequence takes the general form
$\{1, 1, k + (t+1), k^2 + (t+1)k + (t+1), k^3 + (t+1)k^2 + (t+2)k + (2t+1), \ldots\}$
If $k = 1$, this sequence takes the form

$\{1, 1, 2 + t, 3 + 2t, 5 + 4t, 8 + 7t, 13 + 12t, 21 + 20t, \ldots\}$ which can be considered a generalization of Leonardo sequence. Later, if $t = 1$, the classical Leonardo sequence appears.

For some values of t and specific initial conditions, certain types of these numbers have already been the subject of studies.

In this article we t must be non null because if $t = 0$,

1. If the initial conditions are $T(k, 0, 0) = 0$ and $T(k, 0; 1) = 1$, the numbers $T(k, 0, n)$ are the k–Fibonacci numbers $F_{k,n}$ [7, 8]. In this case, if $k = 1$, the numbers $F_{1,n}$ are the terms of the classical Fibonacci sequence $F = \{F_n\}$ and if $k = 2$, the Pell sequence appears.

2. If the initial conditions are $T(k, 0, 0) = 2$ and $T(k, 0, 1) = k$, we have the k–Lucas numbers $L_{k,n}$ [10]. If $k = 1$ it is the classical Lucas sequence $\{L_n\}$ A000032 and if $k = 2$ it is the Pell–Lucas sequence $\{PL_n\}$ A002203.

With the initial conditions $T(1, t, 0) = 1$ and $T(1, t, 1) = 1$, only the following sequences are indexed in the OEIS.

1. For $t = 1$.
   (a) If $k = 1$, the extended $(1, 1)$–Fibonacci number $T(1, 1, n)$ is called the Leonardo number (see the Introduction) Le_n [11, 12], and the sequence of the Leonardo numbers is $\{1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, \ldots\}$: A001595
   (b) If $k = 2$ it is $\{T(2, 1, n)\} = \{1, 1, 4, 10, 25, 61, 148, 358, 865, \ldots\}$: A033539
2. For $t = 2$
   (a) If $k = 1$ it is $\{T(1, 2, n)\} = \{1, 1, 4, 7, 13, 22, 37, 61, 100, 163, 265, \ldots\}$: A111314
   (b) If $k = 2$, then $\{T(2, 2, n)\} = \{1, 1, 5, 13, 33, 81, 197, 477, 1153, 2785, \ldots\}$: 100227

Since the first two addends of the recurrence equation above (Equation (2.1))are exactly the same ones that define the recurrence relation of the k–Fibonacci numbers, we will call these numbers the extended (k, t)–Fibonacci numbers.

However, and without taking into account the complementary term t, there is a difference between both that resides in the initial conditions since while in the k–Fibonacci numbers are $F_{k,0} = 0$ and $F_{k,1} = 1$ in the extended (k, t)–Fibonacci numbers are $T(k, t, 0) = 1$ and $T(k, t, 1) = 1$. Therefore, even for $t = 0$, there is a difference between both values since it is $T(k, 0, n) = F_{k,n-1}$, which indicates that there is a shift in both sequences . If $t \neq 0$, the difference is total between them.

In fact, $T(k, t, n)$ is the sum of a polynomial $P(k) = F_{k,n-1} + F_{k,n}$ of degree n plus the product of t times another polynomial $Q(k) = \dfrac{F_{k,n-1} + F_{k,n} - 1}{k}$ of degree $n - 1$, as we wiil see in the following theorem.

In order to simplify the writing and as long as there is no room for confusion, we will represent the elements $T(k, t, n)$ as T_n.

Theorem 2.1. *If T_n verifies the recurrence relation (2.1), then*

$$T_n = \frac{(k+t)(F_{k,n} + F_{k,n-1}) - t}{k} \tag{2.2}$$

We will prove this theorem by induction, first proving it for $n = 0, 1, 2$ and later extending it to n. Remember that $F_{k,-n} = (-1)^{n+1} F_{k,n}$ so $F_{k,-1} = Fk, 1 = 1$.

$$\begin{aligned}
T_0 &= \frac{(k+t)(F_{k,0} + F_{k,-1}) - t}{k} = \frac{(k+t) - t}{k} = 1 \\
T_1 &= \frac{(k+t)(F_{k,1} + F_{k,0}) - t}{k} = \frac{(k+t) - t}{k} = 1 \\
T_2 &= \frac{(k+t)(F_{k,2} + F_{k,1}) - t}{k} = \frac{(k+t)(k+1) - t}{k} = \frac{k^2 + t\,k + k + t - t}{k} \\
&= k + t + 1
\end{aligned}$$

Suppose the formula (2.2) is true up to T_{n-1}. Then

$$\begin{aligned}
T_{n-1} &= \frac{(k+t)(F_{k,n-1} + F_{k,n-2}) - t}{k} \\
T_{n-2} &= \frac{(k+t)(F_{k,n-1} + F_{k,n-2}) - t}{k} \\
&k\, T_{n-1} + T_{n-2} + t \\
&= \frac{1}{k}\left[(k+t)(k\, F_{k,n-1} + k\, F_{k,n-2} + F_{k,n-2} + F_{k,n-3} - k\,t - t\right] + t \\
&= \frac{1}{k}\left((k+t)(F_{k,n} + F_{k,n+1}) - t\right) = T_n
\end{aligned}$$

If $k = 1$, the k–Fibonacci numbers are the classical Fibonacci fibonacci numbers $\{0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$ and formula (2.2) becomes $T_n = (1+t)(F_n + F_{n-1}) - t = (1+t)F_{n+1} - t$. In this case, we obtain the generalized Leonardo sequence $\{Le_n(t)\} = \{1, 1, 2+t, 3+2t, 5+4t, 8+7t, 13+12t, 21+20t, \ldots\}$, which results in the classical Leonardo sequence if $t = 1$.

If $t = 0$, the formula (2.1) is the recurrence relation that defines the defines the Fibonacci numbers but with different initial conditions.

But if $t = 1$, the relation (2.1) defines the Leonardo numbers [12, 11] and then $Le_n = \frac{2(F_n + F_{n-1}) - 1}{1} = 2F_{n+1} - 1$

Proposition 2.1. *The non-homogeneous recurrence relation (2.1) can be transformed in a homogeneous recurrence relation by the formula*

$$T_n = (k+1)T_{n-1} - (k-1)T_{n-2} - T_{n-3} \tag{2.3}$$

for $n \geq 3$ and initial conditions $T_0 = 1$, $T_1 = 1$, $T_2 = k + t + 1$

Proof. From Equation (2.2)

$$\begin{cases} (k+1)T_{n-1} = \dfrac{k+1}{k}((k+t)(F_{k,n-1}+F_{k,n-2})-t) \\ -((k-1)T_{n-2} = \dfrac{-(k-1)}{k}((k+t)(F_{k,n-2}+F_{k,n-2})-t) \\ -T_{n-3} = \dfrac{-1}{k}((k+t)(F_{k,n-3}+F_{k,n-4})-t) \end{cases}$$

from where

$$\begin{aligned} &(k+1)T_{n-1}-(k-1)T_{n-2}-T_{n-3} = \\ &\frac{k+t}{k}\left((k+1)F_{k,n-1}-(k-1)F_{k,n-2}-F_{k,n-3}\right) \\ &+\frac{k+t}{k}((k+1)F_{k,n-2}-(k-1)F_{k,n-3}-F_{k,n-4}) \\ &-\frac{t}{k}((k+1)-(k-1)-1) \\ &= \frac{k+t}{k}(k\,F_{k,n-1}+F_{k,n-1}-k\,F_{k,n-2}+F_{k,n-3}-F_{k,n-3}) \\ &+\frac{k+t}{k}(k\,F_{k,n-2}+F_{k,n-2}-k\,F_{k,n-3}+F_{k,n-3}-F_{k,n-4})-\frac{t}{k} \\ &\frac{k+t}{k}((k\,F_{k,n-11}+F_{k,n-2})+(k\,F_{k,n-2}+F_{k,n-3}))-\frac{t}{k} \\ &= \frac{1}{k}((k+t)(F_{k,n}+F_{k,n-1})-t) = T_n \end{aligned}$$

If $k=1$, this is the recurrence relation between the generalizaed Leonardo numbers: $Le_n(t) = (1+t)Le_{n-1}(t) - Le_{n-3}(t)$ with initial conditions $Le_0(t)=1$, $Le_1(t)=1$, $Le_2(t)=2+t$

Theorem 2.2. *The formula (2.2) can also be proven from this equation, as we demonstrate below.*

Proof. The characteristic equation of the relation (2.3) is $r^3-(k+1)r^2+(k-1)r+1=0$ and its solutions are $r_1=1$, $r_2=\sigma_1=\dfrac{k+\sqrt{k^2+4}}{2}$ and $r_3=\sigma_2=\dfrac{k-\sqrt{k^2+4}}{2}$. Therefore, the general term will have the form $T_n = C_1+C_2\sigma_1^n+C_3\sigma_2^n$ with the conditions $T_0=1$, $T_1=1$ and $T_2=k+(t+1)$.
So, we have the linear system

$$\begin{aligned} T_0 &= C_1+C_2+C_3=1 \\ T_1 &= C_1+C_2\,\sigma_1+C_3\,\sigma_2=1 \\ T_2 &= C_1+C_2\,\sigma_1^2+C_3\,\sigma_2^2=k+(t+1) \end{aligned}$$

Solving this system of equations we find the values $C_0=-\dfrac{t}{k}$, $C_1=\dfrac{1}{k}\dfrac{1-\sigma_2}{\sigma_1-\sigma_2}(k+t)$ and $C_2=\dfrac{1}{k}\dfrac{-1+\sigma_1}{\sigma_1-\sigma_2}(k+t)$. And taking into account that $(1-\sigma_2)\sigma_1^n=\sigma_1^n+\sigma_1^{n-1}$, $(-1+\sigma_1)\sigma_2^n=-\sigma_2^n-\sigma_2^{n-1}$ and the Binet Identity (1.1) $F_{k,r}=\dfrac{\sigma_1^r-\sigma_2^r}{\sigma_1-\sigma_2}$, finally the formula (2.2) $T_n=\dfrac{1}{k}(-t+(k+t)(F_{k,n}+F_{k,n-1}))$ results.

For now, we therefore have three different ways to find the numbers T_n:

- Through the recurrence relation (2.1): $T_n = k\,T_{n-1}+T_{n-2}+t$
- Using the formula (2.2): $T_n=\dfrac{(k+t)(F_{k,n}+F_{k,n-1})-t}{k}$
- By the second recurrence relation (2.3): $T_n=(k+1)T_{n-1}-(k-1)T_{n-2}-T_{n-3}$

The first recurrence relation needs the two initial conditions $T_0=1$, $T_1=1$ and the second recurrence relations needs a third condition $T_2=k+(t+1)$

For the second formula, it is necessary to use the Binet Identity (1.1).

You can also use a combinatorial formula like $F_{k,n} = \sum_{j=0} \binom{n-1-j}{j} k^{n-1-2j}$ [8].

For the sequence A033539 ($k = 2,\ t = 1$), it is $T(2,1;n) = 3T(2,1;n-1) - T(2,1;n-2) - T(2,1;n-3)$.
A case that we will study in more depth is that of Leonardo numbers.

The recurrence relation (2.3) for $k = 1$ and $t = 1$ $Le_n = 2Le_{n-1} - Le_{n-3}$, so the characteristic equation is $r^3 - 2r^2 + 1 = 0$ whose solutions are $\{1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$, being $\frac{1+\sqrt{5}}{2}$ the golden ratio ϕ and $\frac{1-\sqrt{5}}{2} = 1+\psi = -\frac{1}{\phi}$. So, the general term Le_n will be $Le_n = C_0 + C_1\phi^n + C_2\psi^n$. Solving the system $\{Le_0, Le_1, Le_2\} = \{1, 1, 3\}$, we obtain the coefficients $C_0 = -1,\ C_1 = \frac{1+\sqrt{5}}{\sqrt{5}}, -\frac{1-\sqrt{5}}{\sqrt{5}}$.

Finally the Binet formula for the Leonardo numbers results: $Le_n = -1 + \frac{2}{\sqrt{5}}\left(\phi^{n+1} - \psi^{n+1}\right)$. That is $Le_n = 2F_{n+1} - 1$

Lemma 2.3. *Just as the formula (2.2) indicates that T_n can be expressed in terms of the $F_{k,n}$, it is also possible to express $F_{k,n}$ in terms of the T_n as indicated in the following formula.*

$$F_{k,n} = \frac{T_{n+1} - T_n}{k+t} \tag{2.4}$$

Proof.

$$\begin{aligned}
T_n &= \frac{(k+t)F_{k,n} + F_{k,n-1}) - t}{k} \\
T_{n+1} &= \frac{(k+t)F_{k,n+1} + F_{k,n}) - t}{k} \to F_{k,n} = \frac{k\,T_{k,n+1} + t}{k+t} - F_{k,n+1} \\
T_n &= \frac{(k+t)\left(\frac{k\,T_{k,n+1} + t}{k+t} - F_{k,n+1} + F_{k,n-1}\right) - t}{k} \\
&= \frac{k\,T_{n+1} + t - (k+t)\,k\,F_{k,n} - t}{k} = T_{n+1} - (k+t)F_{k,n} \\
&\to \quad F_{k,n} = \frac{T_{n+1} - T_n}{k+t}
\end{aligned}$$

It is interesting to note that although the parameter t is included in the second member, in reality the formula is independent of its value since $F_{k,n}$ does not depend on t.

2.1 Generating function

The generating function of the sequence of the extended (k,t)–Fibonacci numbers T_n is

$$f(k,t,x) = \frac{1 - k\,x - (1-k-t)x^2}{(1-x)(1-k\,x-x^2)} \tag{2.5}$$

Wee will use the homgeneous recurrence relation (2.3).

$$\begin{aligned}
f(k,t,x) &= T_0 + T_1x + T_2x^2 + T_3x^3 + \cdots \\
&= T_0 + T_1x + T_2x^2 \\
&\quad +((k+1)T_2 - (k-1)T_1 - T_0))x^3 \\
&\quad +((k+1)T_3 - (k-1)T_2 - T_1))x^4 \\
&\quad +((k+1)T_4 - (k-1)T_3 - T_2))x^5 + \cdots \\
&= T_0 + T_1x + T_2x^2 \\
&\quad +(k+1)(T_2x^2 + T_3x^3 + T_4x^4 + \cdots)x \\
&\quad -(k-1)(T_1x + T_2x^2 + T_3x^3 + \cdots)x^2 \\
&\quad -(T_0 + T_1x + T_2x^2 + \cdots)x^3 \\
&= T_0 + T_1x + T_2x^2 + (k+1)(f(k,t,x) - T_1x - T_0)x \\
&\quad -(k-1)(f(k,t,x) - T_0)x^2 - f(k,t,x)x^3 \\
&= T_0 + T_1x + T_2x^2 - (T_1x + T_0)(k+1)x + T_0(k-1)x^2 \\
&\quad +f(x)\left((k+1)x - (k-1)x^2 - x^3\right)
\end{aligned}$$

because all the remaining addends are null. So

$$\begin{aligned}
&\left(1 - (k+1)x + (k-1)x^2 + x^3\right) f(k,t,x) \\
&= 1 + x + (k+t+1)x^2 - (x+1)(k+1)x + (k-1)x^2 \\
&= 1 - k\,x + (t+k-1)x^2 \\
f(k,t,x) &= \frac{1 - k\,x + (t+k-1)x^2}{(1-x)(1-k\,x-x^2)}
\end{aligned}$$

En particular, for the generalized Leonardo numbers $Le_n(t)$, the generating function is $l(t,x) = \dfrac{1-x+t\,x^2}{(1-x)(1-x-x^2)}$ and for the classical Leonardo numbers $l(x) = \dfrac{1-x+x^2}{(1-x)(1-x-x^2)}$

Remark 2.1. There is a direct relationship between the equations (2.5) and (2.3) because the denominator $1-(k+1)x+(k-1)x^2+x^3$ of the generating function determines the recurrence relation indicated in Formula (2.3).

Corollary 2.4. *1. For $k = 1$*

(a) If $t = 0$, it results the generating function of the classical Fibonacci numbers (without F_0) $f(1,0,x) = \dfrac{1}{1-x-x^2}$.

(b) If $t = 1$, $f(1,1,x) = \dfrac{1-x+x^2}{(1-x)(1-x-x^2)}$ is the generating function of the Leonardo sequence A001595.

(c) If $t = 2$, $f(1,2,x) = \dfrac{1-x+2x^2}{(1-x)(1-x-x^2)}$ is the generating function of the sequence A111314.

2. For $k = 2$:

(a) If $t = 1$, $f(2,1,x) = \dfrac{1-2x+2x^2}{1-2x-x^2}$ is the generating function of the sequence A0033539.

(b) *If* $t=2$, $f(2,2,x)=\dfrac{1-2x+3x^2}{(1-x)(1-2x-x^2)}$ *generates the sequence A1002275.*

3. *For* $k=3$ *if* $t=1$, $f(3,1,x)=\dfrac{1-3x+3x^2}{(1-x)(1-3x-x^2)}$ *is the generating function of the sequence A033538*

2.2 Use of the binet identity of the k–fibonacci numbers for the extended (k,t)–Fibonacci numbers

If the Binet Identity for the k–Fibonacci numbers (1.1) is substituted in Equation (2.2), $T(k,t,n)=\dfrac{(k+t)(F_{k,n}+F_{k,n-1})-t}{k}$, the Binet formula for the extended (k,t)–Fibonacci numbers results:

$$T(k,t,n)=\frac{k+t}{k}\left(\frac{\sigma_1^n-\sigma_2^n}{\sigma_1-\sigma_2}+\frac{\sigma_1^{n-1}-\sigma_2^{n-1}}{\sigma_1-\sigma_2}\right)-\frac{t}{k} \tag{2.6}$$

You can also use the combinatorial formula indicated in the introduction for the k–Fibonacci numbers and then

$$T(k,t,n)=\frac{k+t}{k}\left(\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-j}{j}k^{n-1-2j}+\sum_{j=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-2-j}{j}k^{n-2-2j}\right)-\frac{t}{k}$$

For $k=t=1$, the Leonardo numbers results and then

$$Le_n=2\left(\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-j}{j}+\sum_{j=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-2-j}{j}\right)-1$$

3 Conclusions

In this article we have defined a new type of numerical sequences that link the k–Fibonacci numbers with the Leonardo numbers. We have found some properties of these numbers such as the recurrence relation of their elements as well as their generating function. We have also demonstrated the connection between these numbers with the k–Fibonacci numbers.

This opens a new field where we can investigate and establish connections with other parts of mathematics or even science in general.

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Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Competing Interests

Author has declared that no competing interests exist.

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