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Two-Commodity Markovian Inventory System with Compliment and Retrial Demand

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Authors' contributions

This work was carried out in collaboration between all authors. Author NA defined the mathematical model and derived the steady state distributions. Author KJ derived the system performance measures and Numerical illustrations. All authors read and approved the final manuscript.

Research Article

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Abstract

In this article, we consider a stochastic inventory system with two different items in stock, one is major item (I- commodity) and other is gift item (II- commodity). The maximum storage capacity for the i th commodity is $S_i (i = 1,2)$. The demand time points for each commodity are assumed to form a independent Poisson processes. The second commodity is supplied as a gift whenever the demand occurs for the first commodity, but no major item is provided as a gift for demanding a second commodity. (s_1, Q_1) type control policy for the first commodity, with random lead time but instantaneous replenishment for the second commodity are considered. If the inventory position of first commodity (major item) is zero then any arriving primary demand for the first commodity enters into an orbit of finite size N . These orbiting customers compete for service by sending out signals that are exponentially distributed. The joint probability distribution for both commodities and the number of demands in the orbit, is obtained in the steady state case. Various system performance measures in the steady state are derived. The results are illustrated with numerical examples.

Keywords: Compliment item, continuous review, retrial demand, stochastic lead time, markov process.

1 Introduction

Investigations on multi-item / multi-echelon systems are much less compared to those on single commodity systems. The determination of optimal policies and the problems related to a multi-item / multi-echelon systems are, to some extent, dealt by Veinott and Wagner [19] and Veinott

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[20]. Sivazlian [16] discussed the stationary characteristics of a multicommodity single period inventory system. Sivazlian and Stanfel [17] analyzed a two commodity single period inventory system. Kalpakam and Arivarigan [9] analyzed a multi-item inventory model with renewal demands under a joint replenishment policy. They assumed instantaneous supply of items and obtain various operational characteristics and also an expression for the long run total expected cost rate. Krishnamoorthy et.al., [10] analyzed a two commodity continuous review inventory system with zero lead time. A two commodity problem with Markov shift in demand for the type of commodity required, is considered by Krishnamoorthy and Varghese [11]. They obtain a characterization for limiting probability distribution to be uniform. Dhandra and Prasad [8] analyzed a two commodity inventory model in which item 1 is substitutable for item 2 but not vice versa. Associated optimization problems were discussed in all these cases. However in all these cases zero lead time is assumed.

Anbazhagan and Arivarignan [1,2,3] have analyzed two commodity inventory system under various ordering policies. Yadavalli et. al., [21] have analyzed a model with joint ordering policy and varying order quantities. Yadavalli et. al., [22] have considered a two commodity substitutable inventory system with Poisson demands and arbitrarily distributed lead time. In production inventory model Mohebbi [13] has considered a limited-capacity production inventory system with linear production rate and compound Poisson demand processes are subject to independently and randomly changing environmental conditions. Mohebbi [14] has considered production control model for a facility with limited storage capacity in a random environment. The author has assumed that the inventory level of a storage facility which has a limited storage capacity. The sojourn time of each state is exponentially distributed. All stock outs, including the excess demand when a batch size is larger than the inventory level are considered to be lost. In a very recent paper, Anbazhagan et. al. [4] considered analysis of two commodity inventory system with compliment for bulk demand in which, one of the items (say c_1) is designated as the major item and the other items (say c_2) as the sub-item (gift item). The sub-item is supplied whenever the demand for major item is greater than or equal to r (a specified number $>$ 0). (s, S) type policy for the major item, with random lead time but instantaneous replenishment for the gift item are considered. The lost sales for major item is also assumed when the items are out of stock.

All the above models considered the inventory system assumed lost sales of demands occur during stock out periods. Traditionally the inventory models incorporate the stream of customers (either at fixed time intervals or random intervals of time) whose demands are satisfied by the items from the stock and those demands which cannot be satisfied are either backlogged or lost. But in recent times due to the changes in Business environments in terms of technology such as Internet, the customer may retry for his requirements at random time points.

The first study on inventory models with positive lead time, with unsatisfied customers thus created going to an orbit to try again for inventory from there, was by Artelijo et. al. [5]. Whereas their approach is algorithmic, Ushakumari [18] produces analytical solution to the same model. Following these, a number of papers on inventory models with retrial of unsatisfied customers emerged. Krishnamoorthy and Jose [12] analyzed and comparison of inventory systems with service, positive lead time, loss, and retrial of customers. The basic retrial queues were generalized in various directions by a large number of publications. Surveys and Bibliographies of these works are given in [6,7]. The discussion in Sivakumar [15] is on a two-commodity system where customers, encountering both commodities out-of-stock, proceed to an orbit of infinite capacity.

In the present paper, we assumed that both commodities C_1 and C_2 are sold separately and as a gift for each major item a customer buys, a sub-item is supplied with, such as television and stabilizer, refrigerator and stabilizer and so on. It's also assumed that a customer enters into the orbit when the first commodity is out of stock. The remainder of this paper is arranged as follows. In section 2 we describe the mathematical model. In section 3 analysis part of the system. In section 4 is devoted to some system performance measures like the expected waiting time of an orbital customer. Finally in section 5 the total expected cost rate is computed and we provide some results of the numerical experiments carried out for analyzing different aspects of the system under study.

2 Mathematical Model

We consider a stochastic inventory system with two different items in stock, one is major item (I-commodity) and the other is gift item (II-commodity). The maximum storage capacity for the i th commodity is S_i $(i = 1,2)$. The demand time points for each commodity are assumed to form independent Poisson processes each with parameter λ_i $(i = 1,2)$. If the demand occurs for first commodity then one unit of second commodity is supplied as a gift to the customer who ordered a unit of first commodity, but not vice versa i.e., no first commodity is supplied as a gift for ordering a second commodity. As and when the on-hand inventory level of first commodity drops to a prefixed level, $s_1(\geq 0)$, an order for $Q_1(= S_1 - s_1 > s_1)$ units is placed. The lead time of this order is exponentially distributed with parameter $\mu(> 0)$. An $(0, S_2)$ ordering policy is adopted for the second commodity with zero lead time. Since the lead time is zero, the inventory level of second commodity (gift item) is uniformly distributed in the interval $(0, S_2)$, with probability density function $\frac{1}{S_2}$. If the inventory position of first commodity is zero, thereafter any arriving primary demand for the first commodity enters into the orbit of finite size N. In this article we consider the classical retrial policy. More explicitly, when there are $i \geq 1$ demands in the orbit, a signal is sent out according to an exponential distribution with parameter θ. If the inventory position of first commodity is zero and the orbit is full then any arriving primary demand for first commodity is considered to be lost. We have also assume that the inter demand times between primary demands, lead times and retrial demand times are mutually independent random variables.

3 Analysis

Let $L_1(t)$, $L_2(t)$ and $X(t)$ denote the inventory position of commodity-I, the inventory position of commodity-II and the number of demands in the orbit respectively. From the assumptions made on the input and output processes it can be shown that the triplet $\{(L_1(t), L_2(t), X(t)), t \geq 0\}$ is a continuous time Markov chain with state space by $E = E_1 \times E_2 \times E_3$. To determine the infinitesimal generator A of the Markov process with entries are of the form, $A = (\, p((i_1, i_2, i_3), (j_1, j_2, j_3))\,)$, $(i_1, i_2, i_3), (j_1, j_2, j_3) \in E$.

This Markov process gives raise to the following arguments on state transitions in the vector process $\{(L_1(t), L_2(t), X(t)), t \geq 0\}$.

A primary demand for the first commodity takes the state of the process from (i_1, i_2, i_3) to $(i_1 - 1, i_2 - 1, i_3)$ and the intensity of this transition $p((i_1, i_2, i_3), (i_1 - 1, i_2 - 1, i_3))$ is given by λ_1, $i_1 = 1,2,\ldots,S_1$, $i_2 = 1,2,\ldots,S_2$, $i_3 = 0,1,\ldots,N$.

Any arriving demand for the second commodity takes the state of the process from (i_1, i_2, i_3) to $(i_1, i_2 - 1, i_3)$ and the intensity of this transition $p((i_1, i_2, i_3), (i_1, i_2 - 1, i_3))$ is given by λ_2, $i_1 = 0,1,2,\ldots,S_1$, $i_2 = 1,2,\ldots,S_2$, $i_3 = 0,1,\ldots,N$.
If the inventory position of first commodity is zero then any arriving primary demand for the first commodity enters into the orbit. Hence a transition take place from $(0, i_2, i_3)$ to $(0, i_2, i_3 + 1)$ and the intensity of this transition $p((0, i_2, i_3), (0, i_2, i_3 + 1))$ is given by λ_1, $i_2 = 1,2,\ldots,S_2$, $i_3 = 0,1,\ldots,N-1$.

A retrial demand for the first commodity takes the state of the process from (i_1, i_2, i_3) to $(i_1 - 1, i_2 - 1, i_3 - 1)$ and the intensity of this transition $p((i_1, i_2, i_3), (i_1 - 1, i_2 - 1, i_3 - 1))$ is given by $i_3\theta$, $i_1 = 1,2,\ldots,S_1$, $i_2 = 1,2,\ldots,S_2$, $i_3 = 1,2,\ldots,N$.

A transition from (i_1, i_2, i_3) to $(i_1 + Q_1, i_2, i_3)$, for $i_1 = 0,1,2,\ldots,s_1$ $i_2 = 1,2,\ldots,S_2$, $i_3 = 0,1,\ldots,N$, takes place with intensity μ when a replenishment occurs.

For other transition from (i_1, i_2, i_3) to (j_1, j_2, j_3), except $(i_1, i_2, i_3) \neq (j_1, j_2, j_3)$, the rate is zero.

To obtain the intensity of passage, $p((i_1, i_2, i_3), (j_1, j_2, j_3))$ of state (i_1, i_2, i_3), we note that the entries in any row of this matrix add to zero. Hence the diagonal entry is equal to the negative of the sum of the other entries in that row. More explicitly,

$$p((i_1, i_2, i_3), (i_1, i_2, i_3)) = -\sum_{i_1}\sum_{i_2}\sum_{\substack{i_3 \\ (i_1, i_2, i_3) \neq (j_1, j_2, j_3)}} p((i_1, i_2, i_3), (j_1, j_2, j_3))$$

Hence, we have $p((i_1, i_2, i_3), (j_1, j_2, j_3))$

$$
= \begin{cases}
\lambda_1, & \begin{array}{lll} j_1 = i_1, & j_2 = i_2, & j_3 = i_3 + 1, \\ i_1 = 0, & i_2 \in V_1^{S_2}, & i_3 \in V_0^{N-1}, \\ & \text{or} & \\ j_1 = i_1 - 1, & j_2 = i_2 - 1, & j_3 = i_3, \\ i_1 \in V_1^{S_1}, & i_2 \in V_2^{S_2}, & i_3 \in V_0^{N}, \\ & \text{or} & \\ j_1 = i_1 - 1, & j_2 = S_2, & j_3 = i_3, \\ i_1 \in V_1^{S_1}, & i_2 = 1, & i_3 \in V_0^{N}, \end{array} \\
\lambda_2, & \begin{array}{lll} j_1 = i_1, & j_2 = i_2 - 1, & j_3 = i_3, \\ i_1 \in V_0^{S_1}, & i_2 \in V_2^{S_2}, & i_3 \in V_0^{N}, \\ & \text{or} & \\ j_1 = i_1, & j_2 = S_2, & j_3 = i_3, \\ i_1 \in V_0^{S_1}, & i_2 = 1, & i_3 \in V_0^{N}, \end{array} \\
i_3\theta, & \begin{array}{lll} j_1 = i_1 - 1, & j_2 = i_2 - 1, & j_3 = i_3 - 1, \\ i_1 \in V_1^{S_1}, & i_2 \in V_2^{S_2}, & i_3 \in V_1^{N}, \\ & \text{or} & \\ j_1 = i_1 - 1, & j_2 = S_2, & j_3 = i_3 - 1, \\ i_1 \in V_1^{S_1}, & i_2 = 1, & i_3 \in V_1^{N}, \end{array} \\
\mu, & \begin{array}{lll} j_1 = i_1 + Q_1, & j_2 = i_2, & j_3 = i_3, \\ i_1 \in V_0^{s_1}, & i_2 \in V_1^{S_2}, & i_3 \in V_0^{N}, \end{array} \\
-(\lambda_2 + \mu), & \begin{array}{lll} j_1 = i_1, & j_2 = i_2, & j_3 = i_3, \\ i_1 = 0, & i_2 \in V_1^{S_2}, & i_3 = N, \end{array} \\
-(\lambda_1 + \lambda_2 + \mu), & \begin{array}{lll} j_1 = i_1, & j_2 = i_2, & j_3 = i_3, \\ i_1 = 0, & i_2 \in V_1^{S_2}, & i_3 \in V_0^{N-1}, \end{array} \\
-(\lambda_1 + \lambda_2 + \mu + i_3\theta), & \begin{array}{lll} j_1 = i_1, & j_2 = i_2, & j_3 = i_3, \\ i_1 \in V_1^{S_1}, & i_2 \in V_1^{S_2}, & i_3 \in V_0^{N}, \end{array} \\
-(\lambda_1 + \lambda_2 + i_3\theta), & \begin{array}{lll} j_1 = i_1, & j_2 = i_2, & j_3 = i_3, \\ i_1 \in V_{s_1+1}^{S_1}, & i_2 \in V_1^{S_2}, & i_3 \in V_0^{N}, \end{array} \\
0 & \textit{otherwise}
\end{cases}
$$

The infinitesimal generator A can be conveniently expressed as a block partitioned matrix: $A = (A_{i_1 j_1})$,

where

$$[A]_{i_1 j_1} = \begin{cases} F_0, & j_1 = i_1 - 1, \quad i_1 \in V_1^{S_1} \\ F_1, & j_1 = i_1, \quad i_1 \in V_1^{s_1} \\ F_2, & j_1 = i_1, \quad i_1 \in V_{s_1+1}^{S_1} \\ F_3, & j_1 = i_1, \quad i_1 = 0 \\ C, & j_1 = i_1 + Q_1, \quad i_1 \in V_0^{s_1} \\ \mathbf{0}, & otherwise. \end{cases}$$

$$[F_0]_{i_2 j_2} = \begin{cases} H & j_2 = i_2 - 1, \quad i_2 \in V_2^{S_2} \\ & \quad or \\ & j_2 = S_2, \quad i_2 = 1, \\ \mathbf{0}, & otherwise. \end{cases}$$

$$[F_1]_{i_2 j_2} = \begin{cases} H_0 & j_2 = i_2 - 1, \quad i_2 \in V_2^{S_2} \\ & \quad or \\ & j_2 = S_2, \quad i_2 = 1, \\ G & j_2 = i_2, \quad i_2 \in V_1^{S_2} \\ \mathbf{0}, & otherwise. \end{cases}$$

$$[F_2]_{i_2 j_2} = \begin{cases} H_0 & j_2 = i_2 - 1, \quad i_2 \in V_2^{S_2} \\ & \quad or \\ & j_2 = S_2, \quad i_2 = 1, \\ H_1 & j_2 = i_2, \quad i_2 \in V_1^{S_2} \\ \mathbf{0}, & otherwise. \end{cases}$$

$$[F_3]_{i_2 j_2} = \begin{cases} H_0 & j_2 = i_2 - 1, \quad i_2 \in V_2^{S_2} \\ & \quad or \\ & j_2 = S_2, \qquad i_2 = 1, \\ G_1 & j_2 = i_2, \qquad i_2 \in V_1^{S_2} \\ \mathbf{0}, & otherwise. \end{cases}$$

$$[C]_{i_2 j_2} = \begin{cases} G_2 & j_2 = i_2, \qquad i_2 \in V_1^{S_2} \\ \mathbf{0}, & otherwise. \end{cases}$$

It may be noted that the matrices F_0, F_1, F_2, F_3 and C are square matrices of order $S_2(N+1)$. Also the sub matrices H, H_0, G, H_1, G_1 and G_2 are square matrices of order $(N+1)$.

3.1 Steady State Analysis

It can be seen from the structure of A that the homogeneous Markov process $\{(L_1(t), L_2(t), X(t)) : t \geq 0\}$ on the finite space E is irreducible, aperiodic and persistent non-null. Hence the limiting distribution

$$\pi^{(i_1,i_2,i_3)} = \lim_{t\to\infty} Pr[L_1(t) = i_1, L_2(t) = i_2, X(t) = i_3 \mid L_1(0), L_2(0), X(0)],$$

exists. Let $\mathbf{\Pi} = (\Pi^{(0)}, \Pi^{(1)}, \ldots, \Pi^{(S_1)})$
we partition the vector, $\Pi^{(i_1)}$ into as follows, for $i_1 \geq 0$

$$\Pi^{(i_1)} = (\Pi^{(i_1,1)}, \Pi^{(i_1,2)}, \ldots, \Pi^{(i_1,S_2)}) i_1 = 0,1,2,\ldots,S_1$$

which is partitioned as follows,

$$\Pi^{(i_1,i_2)} = (\pi^{(i_1,i_2,0)}, \pi^{(i_1,i_2,1)}, \ldots, \pi^{(i_1,i_2,N)}), i_1 = 0,1,2,\ldots,S_1; i_2 = 1,2,\ldots,S_2.$$

Then the vector of limiting probabilities $\mathbf{\Pi}$ satisfies

$$\mathbf{\Pi} A = \mathbf{0} \ and \ \mathbf{\Pi} e = 1. \qquad (1)$$

The above equation yields the following set of equations:

$$\Pi^{(i_1)}F_3 + \Pi^{(i_1+1)}F_0 = 0, \qquad i_1 = 0$$

$$\Pi^{(i_1)}F_1 + \Pi^{(i_1+1)}F_0 = 0, \qquad i_1 = 1,2,\dots,s_1$$

$$\Pi^{(i_1)}F_2 + \Pi^{(i_1+1)}F_0 = 0, \qquad i_1 = s_1+1,..,Q_1-1$$

$$\Pi^{(0)}C + \Pi^{(i_1)}F_2 + \Pi^{(i_1+1)}F_0 = 0, \quad i_1 = Q_1 \qquad (*)$$

$$\Pi^{(i_1-Q)}C + \Pi^{(i_1)}F_2 + \Pi^{(i_1+1)}F_0 = 0, \qquad i_1 = Q_1+1,..,S_1-1$$

$$\Pi^{(i_1-Q_1)}C + \Pi^{(i_1)}F_2 = 0, \qquad i_1 = S_1$$

After long simplifications, the above equations, (except (*)), yields

$$\Pi^{(i_1)} = \Pi^{(Q_1)}\Omega_{i_1},\, i_1 = 0,1,\dots,S_1$$

where

$$\Omega_{i_1} = \begin{cases} (-1)^{Q_1-i_1}(F_0F_2^{-1})^{(Q_1-(s_1+1))}(F_0F_1^{-1})^{s_1}(F_0F_3^{-1}) & i_1 = 0 \\ (-1)^{Q_1-i_1}(F_0F_2^{-1})^{(Q_1-(s_1+1))}(F_0F_1^{-1})^{((s_1+1)-i_1)} & i_1 = 1,2,\dots,s_1 \\ (-1)^{Q_1-i_1}(F_0F_2^{-1})^{(Q_1-i_1)} & i_1 = s_1+1, s_1+2,\dots,Q_1-1 \\ I, & i_1 = Q_1 \\ (-1)^{((2Q_1+1)-i_1)}\left(\sum_{j=0}^{S_1-i_1}(F_0F_2^{-1})^{((S_1+s_1)-(i_1+j+1))}(F_0F_1^{-1})^{(j+1)}(CF_2^{-1})\right) & \\ & i_1 = Q_1+1, Q_1+2,\dots,S_1 \end{cases}$$

$\Pi^{(Q_1)}$ can be obtained by solving equation (*) $\;and\;\; \mathbf{\Pi}e = 1$.

that is,

$$\Pi^{(Q_1)}\Big((-1)^{Q_1}\Big[(F_0F_2^{-1})^{(Q_1-(s_1+1))}(F_0F_1^{-1})^{s_1}(F_0F_3^{-1})C\Big] + F_2 + (-1)^{Q_1}\left[\sum_{j=0}^{s_1-1}(F_0F_2^{-1})^{(2(s_1-1)-j)}(F_0F_1^{-1})^{(j+1)})(CF_2^{-1})F_0\right]\Big) = 0,$$

and

$$\Pi^{(Q_1)}\Big[(-1)^{Q_1}(F_0F_2^{-1})^{(Q_1-(s_1+1))}(F_0F_1^{-1})^{s_1}(F_0F_3^{-1})+$$

$$\sum_{i_1=1}^{s_1}(-1)^{Q_1-i_1}(F_0F_2^{-1})^{(Q_1-(s_1+1))}(F_0F_1^{-1})^{((s_1+1)-i_1)}$$

$$+\sum_{i_1=s_1+1}^{Q_1-1}(-1)^{Q_1-i_1}(F_0F_2^{-1})^{(Q_1-i_1)}+I+$$

$$\sum_{i_1=Q_1+1}^{S_1}(-1)^{(2Q_1+1-i_1)}\left(\sum_{j=0}^{S_1-i_1}(F_0F_2^{-1})^{(S_1+s_1-(i_1+j+1))}(F_0F_1^{-1})^{(j+1)}(CF_2^{-1})\right)\Big]e=1.$$

4 System Performance Measures

In this section we derive some performance measures of the system under consideration in the steady state.

4.1 Expected Inventory Levels

Let η_1 and η_2 denote the average inventory level for the first commodity and the second commodity respectively in the steady state. Then we have,

$$\eta_1=\sum_{i_1=1}^{S_1}\sum_{i_2=1}^{S_2}\sum_{i_3=0}^{N}i_1\pi^{(i_1,i_2,i_3)} \tag{2}$$

and

$$\eta_2=\sum_{i_1=0}^{S_1}\sum_{i_2=1}^{S_2}\sum_{i_3=0}^{N}i_2\pi^{(i_1,i_2,i_3)} \tag{3}$$

4.2 Expected Reorder Rates

Let η_3 and η_4 denote the expected reorder rate for the first and second commodities respectively. Then we have,

$$\eta_3=\sum_{i_2=1}^{S_2}\sum_{i_3=0}^{N}(\lambda_1+i_3\theta)\pi^{(s_1+1,i_2,i_3)} \tag{4}$$

and

$$\eta_4=\sum_{i_1=1}^{S_1}\sum_{i_3=0}^{N}(\lambda_1+\lambda_2+i_3\theta)\pi^{(i_1,1,i_3)}+\sum_{i_3=0}^{N}\lambda_2\pi^{(0,1,i_3)} \tag{5}$$

4.3 Expected Balking Rate

Let η_5 denote the expected balking rate. Then we have,

$$\eta_5 = \sum_{i_2=1}^{S_2} \lambda_1 \pi^{(0,i_2,N)} \tag{6}$$

4.4 Expected Number of Demands in the Orbit

Let η_6 denote the expected number of demands in the orbit. Then we have

$$\eta_6 = \sum_{i_1=0}^{S_1}\sum_{i_2=1}^{S_2}\sum_{i_3=1}^{N} i_3 \pi^{(i_1,i_2,i_3)} \tag{7}$$

4.5 The Overall Rate of Retrials

Let η_7 denote the overall rate of retrials in the steady state . Then we have

$$\eta_7 = \sum_{i_1=0}^{S_1}\sum_{i_2=1}^{S_2}\sum_{i_3=1}^{N} i_3 \theta \pi^{(i_1,i_2,i_3)} \tag{8}$$

4.6 The Successful Rate of Retrial

Let η_8 denote the successful retrial rate in the steady state . Then we have

$$\eta_8 = \sum_{i_1=1}^{S_1}\sum_{i_2=1}^{S_2}\sum_{i_3=1}^{N} i_3 \theta \pi^{(i_1,i_2,i_3)} \tag{9}$$

4.7 Fraction of Successful Rate of Retrials

Let η_9 denote the fraction of successful retrial rate in the steady state . Then we have

$$\eta_9 = \frac{\eta_8}{\eta_7} \tag{10}$$

5 Cost Analysis

To compute the total expected cost per unit time (total expected cost rate), we consider the following costs.

c_{h1} : The inventory holding cost per unit item per unit time for I-commodity.

c_{h2} : The inventory holding cost per unit item per unit time for II-commodity.

c_{s1} : The setup cost per order for I-commodity.

c_{s2} : The setup cost per order for II-commodity.

c_b : Balking cost per customer per unit time.

c_w : Waiting cost of a orbiting demands per unit time.

The long run total expected cost rate is given by

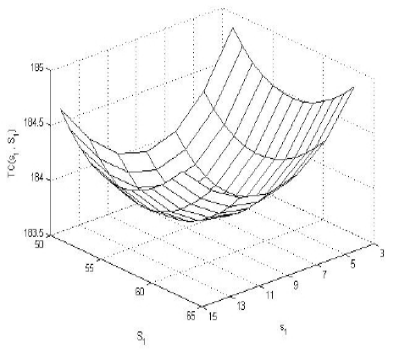
$$TC(s_1, S_1) = c_{h1}\eta_1 + c_{h2}\eta_2 + c_{s1}\eta_3 + c_{s2}\eta_4 + c_b\eta_5 + c_w\eta_6, \quad (11)$$

where $\eta's$ are as given in $(2)-(7)$.

Due to the complex form of the limiting distribution, it is difficult to discuss the properties of the cost function analytically. Hence, a detailed computational study of the cost function is carried out.

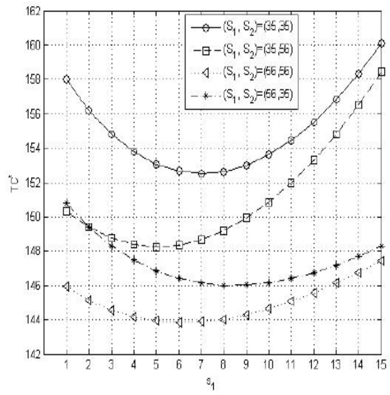
5.1 Numerical Illustrations

To study the behavior of the model developed in this work, several examples were performed and a set of representative results is shown here. Although we have not shown the convexity of $TC(s_1, S_1)$ analytically, our experience with considerable numerical examples indicates the function $TC(s_1, S_1)$, to be convex. We use simple numerical search procedure to get the optimal values of TC, S_1 and s_1 (say TC^*, S_1^* and s_1^*). A typical three dimensional plot of the total expected cost function is given in Figure 1. We have studied the effect of varying the system parameters and costs on the optimal values and the results agreed with what one would expect. Some of our results are presented in Tables 2 through 4, where the upper entries in each cell give the S_1^* and s_1^*, respectively, and the lower entry gives the corresponding TC^*.



$S_2 = 35$, $N = 7$, $\lambda_1 = 11$, $\lambda_2 = 12$, $\mu = 2$, $\theta = 10.5$, $c_{h1} = 1$, $c_{h2} = 2$, $c_{s1} = 65$, $c_{s2} = 79$, $c_b = 3.9$, $c_w = 4.5$

Figure 1: A three dimensional plot of the cost function $TC(s_1, S_1)$



$N = 7$, $\lambda_1 = 11$, $\lambda_2 = 12.9$, $\mu = 2$, $\theta = 10.5$, $c_{h1} = 1$, $c_{h2} = 2$, $c_{s1} = 65$, $c_{s2} = 79$, $c_b = 3.9$, $c_w = 4.5$

Figure 2: Effect of s_1 values on total expected cost rate

Moreover , Figure 1 refers the changes of s_1 and S_1 are how to affect the total expected cost rate. The Table 1, gives the total expected cost rate for various combination of s_1 and S_1 when fixed values for other parameters and costs are assumed. They are $S_2 = 35, N = 7, \lambda_1 = 11, \lambda_2 = 12.9, \mu = 2, \theta = 10.5, c_{h1} = 1, c_{h2} = 2,$
$c_{s1} = 65, c_{s2} = 79, c_b = 3.9,\ c_w = 4.5.$
The values of $TC(s_1, S_1)$ are given in the Table 1. The optimal cost for each S_1 is shown in bold and the optimal cost for each s_1 is underline. The numerical values shows that $TC(s_1, S_1)$ is a convex function in (s_1, S_1) and the (possibly local) optimum occurs at $s_1 = 12$, and $S_1 = 56$.

Table 1. Total expected cost rate as a function of s_1 and S_1

S_1 / s_1	**54**	**55**	**56**	**57**	**58**
10	184.094164	184.094164	183.741836	183.806828	184.008779
11	184.052253	183.792822	183.703936	183.765619	183.960906
12	184.033740	183.778518	183.690359	183.749469	183.939019
13	184.037141	183.786443	183.699492	183.756677	183.941314
14	184.061097	183.815188	183.729860	183.785693	183.966148

Example 5.2

Here, we study the impact of the arrival rate of first commodity, λ_1, the arrival rate of second commodity, λ_2, the lead time rate of first commodity, μ, and the retrial rate, θ, on the optimal values (s_1^*, S_1^*) and the corresponding total expected cost rate TC^* towards this end, we first fix the cost values as $c_{h1} = 1,\ c_{h2} = 2,\ c_{s1} = 65,\ c_{s2} = 79,\ c_b = 3.9$ and $c_w = 4.5$.

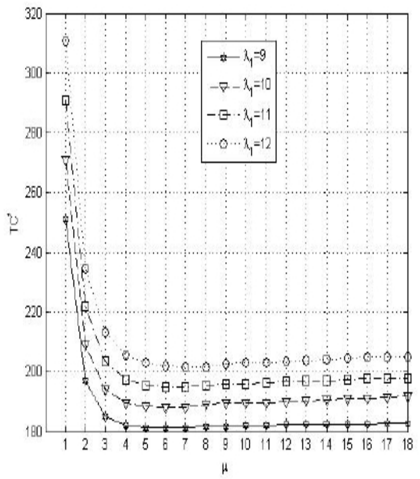
We observe the following from Table 2:

If any one of the parameters λ_1, λ_2 and θ is increases monotonically and other parameters are fixed then the total expected cost rate increases, S_1^* and s_1^* increase. This is to be expected since the arrivals occurs closer and closer, the optimal reorder level, lost sales and the maximum inventory level increase to prevent more waiting time of the customers in the orbit.

As is to be expected as μ increases monotonically and other parameters are fixed then TC^*, S_1^* and s_1^* decrease.

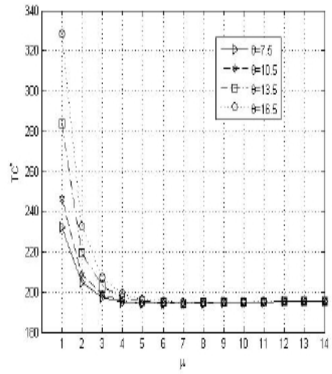
Table 2. Effect parameters on the optimal values S_2=32, N=7

λ_2		**12**			**12.9**			**13.8**		
θ		**10**	**10.5**	**11**	**10**	**10.5**	**11**	**10**	**10.5**	**11**
λ_1	μ									
	1.8	54 12	54 12	55 12	54 12	55 12	55 12	54 12	55 12	55 12
		185.822701	186.000605	186.175835	190.648248	190.825326	190.999404	195.473795	195.648895	195.822972
10		52 11	53 11	53 11	53 11	53 11	53 11	53 11	53 11	53 11
	2	184.501099	184.650761	184.749432	189.317972	189.466369	189.615040	194.133580	194.281977	194.430648
		51 10	51 10	51 10	51 10	51 10	51 10	51 10	51 10	51 10
	2.2	183.410622	183.543640	183.676929	188.222292	188.355310	188.488600	193.033962	193.166980	193.300270
	1.8	58 13	58 13	58 13	58 13	58 13	58 13	58 13	58 13	58 14
11		193.988097	194.182497	194.377220	198.817268	199.011668	199.206391	203.646439	203.840839	204.025819
		56 12	56 12	56 12	56 12	56 12	56 12	56 12	56 12	56 12
	2	192.555790	192.719727	192.883966	197.375189	197.539128	197.703365	202.194588	202.358525	202.522764
	2.2	54 11	54 11	54 11	54 11	54 11	54 11	54 11	54 11	54 11
		191.380745	191.525412	191.670373	196.194602	196.339269	196.484230	201.008460	201.153126	201.298088
	1.8	61 14	61 14	61 14	61 14	61 14	61 14	61 14	61 14	62 15
12		202.058939	202.273020	202.487389	206.893249	207.107330	207.310619	211.727559	211.941640	212.132708
	2	59 13	59 13	59 13	59 13	59 13	59 13	59 13	59 13	59 13
		200.520680	200.699340	200.878329	205.343526	205.522186	205.701176	210.166373	210.345033	210.524023
	2.2	57 12	57 12	57 12	57 12	57 12	57 12	57 12	57 12	58 12
		199.257106	199.419285	199.575202	204.079523	204.235124	204.391042	208.895362	209.050964	209.205611



$S_1 = 56$, $S_2 = 35$, $s_1 = 12$, $N = 7$, $\lambda_2 = 12.9$, $\theta = 10.5$, $c_{h1} = 1$, $c_{h2} = 2$, $c_{s1} = 65$, $c_{s2} = 79$, $c_b = 3.9$, $c_w = 4.5$

Figure 3: TC^* versus μ



$S_1 = 56$, $S_2 = 35$, $s_1 = 12$, $N = 7$, $\lambda_1 = 11$, $\lambda_2 = 12.9$, $c_{h1} = 1$, $c_{h2} = 2$, $c_{s1} = 65$, $c_{s2} = 79$, $c_b = 3.9$, $c_w = 4.5$

Figure 4: TC^* versus μ

Example 5.3

Here, we study the impact of the setup cost of first commodity c_{s1}, holding cost of first commodity c_{h1}, holding cost of second commodity c_{h2}, and waiting cost c_w, on the optimal values (s_1^*, S_1^*) and the corresponding total expected cost rate TC^*. Towards this end, we first fix the parameter values as $\lambda_1 = 11$, $\lambda_2 = 12.9$, $\mu = 2$, and $\theta = 10.5$.

We observe the following from Tables 3 and 4.

As is to be expected, TC^*, S_1^* and s_1^* increase monotonically when c_w increases. This is because if the waiting cost increases then we have to maintain high inventory to reduce the number of waiting customers.

As is to be expected, TC^* and S_1^* increase with c_{s1} increases. Also we notice that s_1^* decreases when c_{s1} increases. This is because if setup cost increases, then we have to maintain high inventory to avoid frequent ordering.

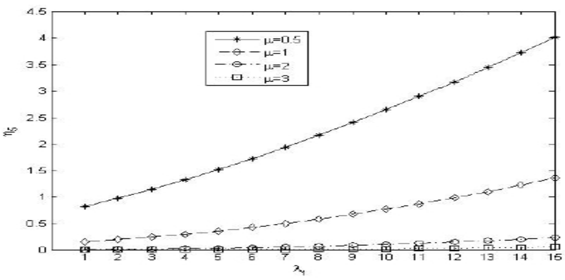
The total expected cost rate increases when c_{h1} and c_{h2} increase. Also we notice that S_1^* and s_1^* decrease when c_{h1} increases.

Table 3. Influence of c_w and c_{s1} on the optimal values

c_w / c_{s1}	4	4.5	5	5.6	6
60	55 12 181.89299	56 12 183.15255	57 13 184.37852	58 13 185.66623	59 14 186.92571
65	55 11 182.43167	56 11 183.69036	57 12 184.91550	58 13 185.65772	59 14 186.00375
70	56 10 182.97036	56 10 184.22817	57 11 185.45247	58 12 187.00722	59 13 189.55551
75	56 10 183.00929	57 10 185.12399	58 9 186.62233	59 9 187.43167	60 8 189.15998
80	57 9 184.55222	57 9 186.00355	58 9 187.65979	59 8 189.45147	59 8 190.97366

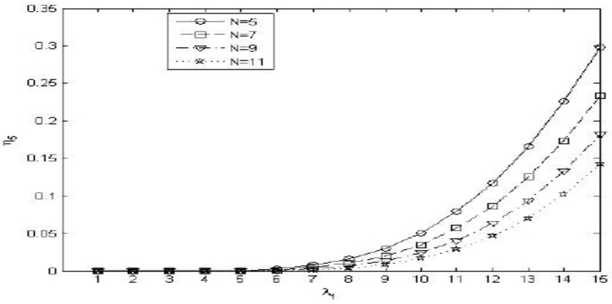
Table 4. Effect of varying c_{h1} and c_{h2} on the optimal values

c_{h1} / c_{h2}	.7	1	1.3	1.6	1.9
1.5	65 13 174.24503	56 12 183.66175	51 11 191.91194	45 10 197.45521	40 8 200.42512
2.0	65 13 174.26554	56 12 183.69036	51 11 191.94908	46 10 197.49321	40 8 202.49132
2.5	65 13 174.28606	56 12 183.73897	51 11 191.98621	46 10 197.53421	39 8 205.00010
3.0	64 13 174.30810	54 12 183.80565	50 11 192.00552	44 10 180.98731	39 8 207.53241
3.5	63 12 174.35232	53 11 183.88565	50 10 192.32001	43 9 183.15782	38 7 209.19832



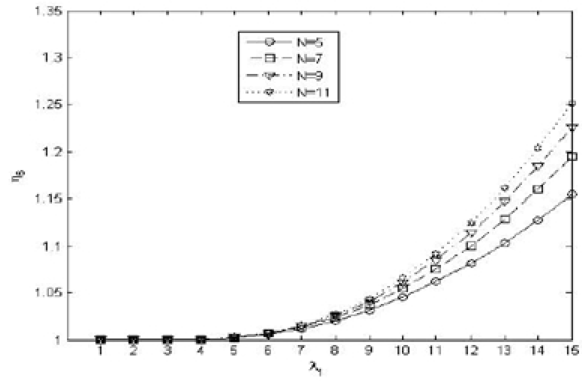
$S_1 = 56$, $S_2 = 35$, $s_1 = 12$, $N = 7$, $\lambda_2 = 12.9$, $\theta = 10.5$, $c_{h1} = 1$, $c_{h2} = 2$, $c_{s1} = 65$, $c_{s2} = 79$, $c_b = 3.9$, $c_w = 4.5$

Figure 5: η_5 versus λ_1



$S_1 = 56$, $S_2 = 35$, $s_1 = 12$, $\lambda_2 = 12.9$, $\theta = 10.5$, $\mu = 2$, $c_{h1} = 1$, $c_{h2} = 2$, $c_{s1} = 65$, $c_{s2} = 79$, $c_b = 3.9$, $c_w = 4.5$

Figure 6: η_5 versus λ_1



$S_1 = 56$, $S_2 = 35$, $s_1 = 12$, $\lambda_2 = 12.9$, $\theta = 10.5$, $c_{h1} = 1$, $c_{h2} = 2$, $c_{s1} = 65$, $c_{s2} = 79$, $c_b = 3.9$, $c_w = 4.5$

Figure 7: η_6 versus λ_1

The four curves in Figure 2 correspond to $(S_1, S_2) = $ (35,35), (35,56), (56,56) and (56,35) represent different convex functions of s_1. The assumed values for the other parameters and costs are shown in the Figure itself.

Figures 3-4 show that the effect of the rate μ, on TC^*. In Figures 3 and 4, we have presented four curves which correspond to $\lambda_1 = 9,10,11,12$ and $\theta =$ 7.5, 10.5, 13.5, 16.5 respectively. From this observation the TC^* has minimum in each curves.

In Figures 5 and 6, the η_5 is plotted against λ_1. In Figures 5 and 6, we have presented four curves which correspond to $\mu =$ 0.5, 1, 2, 3 and $N = 5,7,9,11$ respectively. The balking rate increases when λ_1 increases in both Figures. Moreover, in Figure 5, as μ increases, the η_5 decreases and in Figure 6, as the size of the orbit becomes larger, the balking rate becomes lower. Also, we notice that in Figure 6, as λ_1 increases, η_5 initially stabilizes then it increases.

The expected waiting time for orbiting customers is an increasing function of arrival rate of first commodity (Figure 7) and this behaviour is maintained for various values of N, namely $N = 5$, 7, 9, 11. However, the expected waiting time is higher if N is larger.

6 Conclusion

In this paper, we analyzed a two commodity retrial inventory system with compliment where one item is designated as the major item and the other as the sub item (gift item), such as television and stabilizer, refrigerator and stabilizer and so on. The joint probability distribution of the inventory levels and the number of demands in the orbit in the steady state and the stationary measures of system performances have been derived. Numerical illustration has also been provided to the existence of local optimum when the total expected cost function is treated as a function of two variables s_1 and S_1. The authors are working in the direction of generalizing the demand process as renewal type.

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Competing Interests

Authors have declared that no competing interests exist.

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Appendix

e : a column vector of appropriate dimension containing all ones

0 : Zero matrix

$[A]_{ij}$: entry at $(i,j)^{th}$ position of a matrix A

$k \in V_i^j : k = i, i+1, \ldots j$

$E_1 : \{0,1,2,\ldots,S_1\}$

$E_2 : \{0,1,2,\ldots,S_2\}$

$E_3 : \{0,1,2,\ldots,N\}$

$E : E_1 \times E_2 \times E_3$

In particular, we shall use:

$$[H]_{i_3 j_3} = \begin{cases} \lambda_1, & j_3 = i_3, \quad i_3 \in V_0^N, \\ i_3\theta, & j_3 = i_3 - 1, \quad i_3 \in V_1^N, \\ 0, & otherwise. \end{cases}$$

$$[H_0]_{i_3 j_3} = \begin{cases} \lambda_2 & j_3 = i_3, \quad i_3 \in V_0^N \\ 0, & otherwise, \end{cases}$$

$$[G]_{i_3 j_3} = \begin{cases} -(\lambda_1 + \lambda_2 + \mu + i_3\theta), & j_3 = i_3, \quad i_3 \in V_0^N, \\ 0, & otherwise \;, \end{cases}$$

$$[H_1]_{i_3 j_3} = \begin{cases} -(\lambda_1 + \lambda_2 + i_3\theta), & j_3 = i_3, \quad i_3 \in V_0^N, \\ 0, & otherwise \;, \end{cases}$$

$$[G_1]_{i_3 j_3} = \begin{cases} \lambda_1, & j_3 = j_3 + 1, \quad i_3 \in V_0^{N-1}, \\ -(\lambda_2 + \mu), & j_3 = i_3, \quad i_3 = N, \\ -(\lambda_1 + \lambda_2 + \mu), & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ 0, & otherwise \;, \end{cases}$$

$$[G_2]_{i_3 j_3} = \begin{cases} \mu, & j_3 = j_3, \quad i_3 \in V_0^N, \\ 0, & otherwise \;, \end{cases}$$

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