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Examples of Simply and Multiply Connected Fatou Sets for a Class of Meromorphic Functions

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Research Article

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Abstract

Aims: We give some families which are meromorhic outside a compact countable set B of essential singularities. Our aim is to give some examples of the stable set (called the Fatou set) and the unstable set (called the Julia set) since there is not study of examples of any parametric family of this class of functions (called in the introduction functions of class $\mathcal{K}$) in complex dynamics.
Study design: We study components of the Fatou set and some theorems related with the iteration of functions in class $\mathcal{K}$ and design a computational program to give examples of the Julia and Fatou sets.
Place and Duration of Study: F.C. Físico-Matemáticas, Benemérita Universidad Autónoma de Puebla, México between June 2011 and July 2012.
Methodology: We use some theorems of complex dynamics in order to study components of the Fatou set. We program some algorithms in C and get the picture of this set.
Results: Given a family in class $\mathcal{K}$ we get some mathematical results of the Fatou and Julia sets and its pictures for some parameters given.
Conclusion: Taking some families in class $\mathcal{K} \cap S_k$ we give examples of the Fatou set which can be either simply-connected or multiply-connected in the last case the Julia set is a totally disconnected set.

Keywords: Iteration; Fatou set; Julia set; simply and multiply connected components

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1 Introduction

One of the main ideas in complex dynamics is to divide the plane into the Fatou set and its complement, called the Julia set. For a more detailed description of these sets we refer to [3] and [9].

We are interested in the dynamics of functions which are meromorphic outside a compact countable set of essential singularities. We recall that an essential singularity of a function f of one complex variable is defined as a singularity (isolated) that is neither a pole nor a removable singularity.

We define the following concepts assuming that the function $f(z)$ is a meromorphic function.

Definition 1.1. If there exist z_0 such that $f(z_0) = c$ where $f(z_0) = 0$ (z_0 is called a critical point) we say that c is a critical value.

An example of the above definition can be given by the function $f(z) = \sin z$, where the critical points of the function are $(2k-1)\pi/2$, $k \in \mathbb{Z}$, and the critical values are ± 1.

Definition 1.2. If there exist a curve Γ going to ∞ such that $f(z) \to a$ as $z \to \infty$ along Γ we say that a is an asymptotic value of the function.

An example of an asymptotic value can be given by the function $f(z) = e^z$. We can take the curve $\Gamma : -t+0i$, $t > 0$ and $t \in \mathbb{R}$ going to $-\infty$. Thus $e^z \to 0$ along Γ, this is 0 is an asymptotic value of $f(z)$.

The singular values of a meromorphic function $f(z)$, denoted by SV, are the critical values and the asymptotic values.

The following class of functions was introduced and studied by Bolch in [4, 5, 6].

$$\mathcal{K} = \{f : \widehat{\ } \setminus B \to \widehat{\ } \mid \ B \text{ is a compact countable set and } f \text{ is meromorphic}\}.$$

The set B is formed by the essential singularities of f, where f is non constant, we assume B to have at least two elements and we allow f to have poles. The set B has capacity zero, see [5].

If f is a function in class $\mathcal{K}$ the sequence formed by its iterates is denoted by $f^0 := \mathrm{Id}$, $f^n := f \circ f^{n-1}$, $n \in \mathbb{N}$.

Definition 1.3. We say that z_0 is a fixed periodic point of period $p \in \mathbb{N}$ of a complex function f if $f^P(z_0) = z_0$. When $p = 1$ we say that z_0 is a fixed point.

Definition 1.4. We define the multiplier of a complex function f as $|f'(z)|$. Suppose that z_0 is a fixed point of f when: (i) $|f'(z_0)| < 1$ we say that z_0 is attracting, (ii) $|f'(z_0)| > 1$ we say that z_0 is repelling and (iii) if $|f'(z_0)| = 1$ we say that z_0 is indifferent.

Definition 1.5. The Fatou set $F(f)$ is defined by the set of points $z \in \widehat{\ } \setminus B$ such that the sequence $(f^n)_{n\in\mathbb{N}}$ is well defined and normal in some neighborhood of z. The Julia set is the complement of $F(f)$ and we shall denote it by $J(f)$.

We recall some standard properties of Fatou and Julia sets for f in class $\mathcal{K}$.

(i) $F(f)$ is open and $J(f)$ is closed.
(ii) $J(f)$ is perfect and uncountable.
(iii) $F(f)$ and $J(f)$ are completely invariant in the sense that $f(z)$ if and only if $f(z) \in F(f)$ and $z \in J(f) \setminus B$ if and only if $f(z) \in J(f)$.
(v) $F(f) = F(f^p)$ and $J(f) = J(f^p)$ for every $p \in \mathbb{N}$.
(vi) The Julia set $J(f)$ is the closure of the set repelling periodic points of all orders.

The classification of components in the Fatou set is given as follows.

1. If $f^n(U) \subset U$ for some integer $n \geq 1$, then we call U a periodic component of $F(f)$. The minimum n is the period of the component. In particular, if $n = 1$, then such a component U is said to be an invariant component or a fixed component.
2. If $f^m(U)$ is periodic for some integer $m \geq 0$, we call U a pre-periodic component of $F(f)$. In particular, if U is pre-periodic but not periodic, then we call U a really pre-periodic component.
3. Otherwise, all $\{f^n(U)\}$ are disjointed, and we call U a wandering domain.

If U is periodic component of $F(f)$ of period p, that is if $f^p(U) \subset U$, the classification of the periodic component is given as follows for class $\mathcal{K}$.

(i) If U contains an attracting periodic point z_0 of period p. Then $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$, and U is called the immediate attractive basin of z_0 or attracting component.

(ii) If ∂U contains a periodic point z_0 of period p and $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$. Then $(f^p)^{'}(z_0) = 1$ if $z_0 \in$. (For $z_0 = \infty$ we have $(g^p)^{'}(0) = 1$ where $g(z) = \frac{1}{f(\frac{1}{z})}$.) In this case, U is called a Leau domain or parabolic component.

(iii) If there exists an analytic homeomorphism $\varphi : U \to D$ where D is the unit disc such that $\varphi(f^p(\varphi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then U is called a Siegel disc.

(iv) If there exists an analytic homeomorphism $\varphi : U \to A$ where A is an annulus $A = \{z : 1 < |z| < r\}, r > 1$, such that $\varphi(f^p(\varphi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. Then U is called a Herman ring.

(v) If There exists $z_0 \in \partial U$ such that $f^{np}(z) \to z_0$, for $z \in U$ as $n \to \infty$, but $f^p(z_0)$ is not defined. Then U is called a Baker domain.

In Section 2 we define a subclass of functions $S_{\mathcal{K}}$ of $\mathcal{K}$ and give some examples of the Fatou and Julia sets for functions in this class. This examples of the Fatou set are simply-connected and multiply-connected. Using Theorem 2.3 we compute examples when the Julia set is totally disconnected, this is a Cantor set.

2 Functions in class $\mathcal{K} \cap S_k$

For functions in class $\mathcal{K}$ the components of the Fatou set can be (a) simply connected or (b) multiply connected [5].

We say that a function $f(z)$ in class $\mathcal{K}$ belongs to the class $S_{\mathcal{K}}$ if the set of singular values SV of $f(z)$ is finite. The singular values of a function $f(z)$ in class $\mathcal{K}$ are the critical values and the asymptotic values, see Section 1.

The following results were proved for a more general class of meromorphic functions, see [[2]], since the proofs remain valid for functions in class $\mathcal{K} \cap S_{\mathcal{K}}$ we do not prove them.

Theorem 2.1. *If $f \in \mathcal{K} \cap S_{\mathcal{K}}$, then it has no Baker domains.*

Theorem 2.2. *If $f \in \mathcal{K} \cap S_{\mathcal{K}}$, then it has no wandering domains.*

Theorems 2.1 and 2.2 exclude Baker and wandering domains, thus the possibilities for the components of the Fatou set for functions in class $\mathcal{K} \cap S_{\mathcal{K}}$ are: attracting components, parabolic components, Siegel discs and Herman rings. The following subsections contain examples of the Fatou set which can have simply-connected components or have a multiply-connected component.

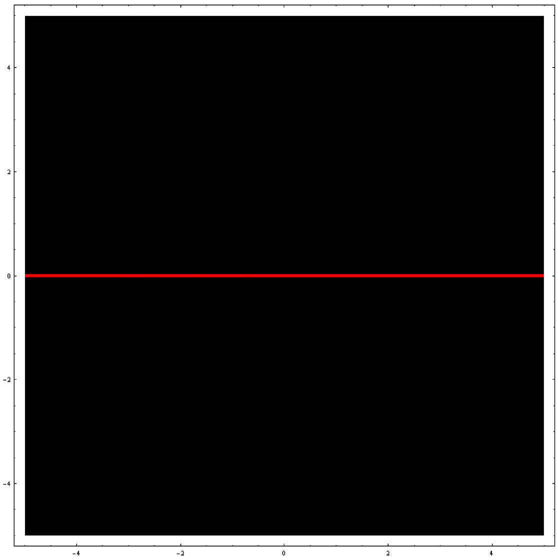

Figure 1: The Julia set is $\mathbb{R}$ and the Fatou set has two components

2.1 Example of a function in class $\mathcal{K} \cap S_k$ for which the components of the Fatou set are simply-connected

Example 1. Take the family $f_{\lambda,\mu}(z) = \tan(\lambda \tan(\mu z))$ where $\lambda, \mu \in$. This family is in class $\mathcal{K}$ since it has a countable compact set of essential singularities when $\cos \mu z = 0$

When $|\lambda\mu| \geq 1$ the asymptotic values of $f_{\lambda,\mu}$ are $\pm\lambda i$ and $\pm\tan(\lambda i)$, thus the set of singular values of the family is finite, therefore the family is in class $\mathcal{K} \cap S_k$.

For this example the Julia set is connected the arguments are similar to those in [8] for transcendental meromorphic functions. This is, the Fatou set has two different completely invariant components, say H^+ and H^-, each of them containing the singular values. The components H^+, H^- are separated by $J(f) = \mathbb{R}$. Thus the components of the Fatou set are simply-connected, see Figure 1 (for $\lambda\mu = 1$).

2.2 Examples of functions in class $\mathcal{K} \cap S_k$ for which the component of the Fatou set is multiply-connected

By using surgery construction, see [7], it is possible to construct examples of functions in class $\mathcal{K}$ which have a bounded Herman ring, thus for this examples the Fatou set is multiply-connected.

We want to give some strong conditions for which the Fatou set is multiply-connected. The following theorem gives examples of multiply-connected Fatou components-

Theorem 2.3. *Let $f \in \mathcal{K} \cap S_{\mathcal{K}}$ and suppose that there is an attracting fixed point whose Fatou component H contains all the singular values of f. Then $J(f)$ is totally disconnected.*

The proof of Theorem 2.3 can be deduced from Theorem G in [2].

Example 2. The family $f_{\lambda,\mu}(z) = \tan(\lambda \tan(\mu z))$, $\lambda, \mu \in$, is in class $\mathcal{K} \cap S_k$, see Example 1. The point $z = 0$ is a fixed point of the family since $f_{\lambda,\mu}(0) = \tan(\lambda \tan(\mu 0)) = 0$.

Observe that $|f'_{\lambda,\mu}(0)| = |\lambda\mu|$, so when $|\lambda\mu| < 1$ the point $z = 0$ is an attracting fixed point.

The singular values of the family $\{\pm\lambda i, \pm \tan(\lambda i)\}$ are in an attracting completely invariant component H, see [2], By Theorem 2.3 the family $f_{\lambda,\mu}(z)$ has totally disconnected Julia set for λ, μ small constants. Thus the Fatou set is an attracting completely invariant multiply-connected component.

Example 3. Take the family

$$f_{\lambda,\mu,c}(z) = \lambda e^{\frac{1}{z^2-c}} - \mu, \text{ where } c, \mu \in \text{ with } c \neq 0. \tag{2.1}$$

Observe that the essential singularities of the family in (2.1) are $\pm\sqrt{c}$. The derivative of $f_{\lambda,\mu,c}(z)$ is given by

$$f'_{\lambda,\mu,c}(z) = \lambda e^{\frac{1}{z^2-c}} \frac{-2z}{(z^2-c)^2}. \tag{2.2}$$

The Equation 2.2 can be written as follows.

$$f'_{\lambda,\mu,c}(z) = \lambda e^{\frac{1}{z^2-c}} \frac{-2z}{(z^2-c)^2} = \lambda e^{\frac{1}{z^2-c}} \frac{-2z}{z(z^3 - 2zc + c^2/z)} = \lambda e^{\frac{1}{z^2-c}} \frac{-2}{(z^3 - 2zc + c^2/z)}. \tag{2.3}$$

From Equation 2.2 we have that $f'_{\lambda,\mu,c}(0) = 0$ and from Equation 2.3 we have that $f'_{\lambda,\mu,c}(\infty) = 0$. Thus $0, \infty$ are the critical points and $0, (\lambda - \mu)$ are the critical values of $f_{\lambda,\mu,c}(z)$. The family has an asymptotic value.

Case when $\mu = \lambda e^{-c}$ in the family (2.1)

We study the family in (2.1) when the parameter $\mu = \lambda e^{-1/c}$, $\lambda \in \mathbb{R}$ is very small and $c \in \mathbb{R}^+$, $c \neq 0$. We have the following subfamily.

$$f_{\lambda,c}(z) = \lambda e^{\frac{1}{z^2-c}} - \lambda e^{-1/c}. \tag{2.4}$$

This new family is in class $\mathcal{K}$ since it has two essential singularities at $\pm\sqrt{c}$. The family has a fixed point at $z = 0$, since $f_{\lambda,c}(0) = \lambda e^{\frac{1}{-c}} - \lambda e^{-1/c} = 0$, for any $\lambda, c \in$, $c \neq 0$.

The family $f_{\lambda,c}(z) = \lambda e^{\frac{1}{z^2-c}} - \lambda e^{-1/c}$ has critical values at $\{0, \lambda(1 - e^{\frac{1}{-c}})\}$ and an asymptotic value. Thus the set of singular values of (2.4) is a finite set. Therefore, the family is in class $\mathcal{K} \cap S_k$.

The critical value $z = 0$ is in the Fatou set, since it is an attracting fixed point. The asymptotic value of the family (2.4) is also in the Fatou set. By Theorem 2.3 the Julia set is totally disconnected. Figure 2 shows the Julia set for the parameters $c = 1$, $\lambda = 1/2$. The Fatou set is an attracting completely invariant multiply-connected component.

If we look at functions which are transcendental meromorphic with at least one pole (which is not omitted) and has just an essential singularity at ∞ there are examples differents from the family $\lambda \tan z$ for which the Fatou set is either simply connected or multiply connected. In [1] it was given the following function.

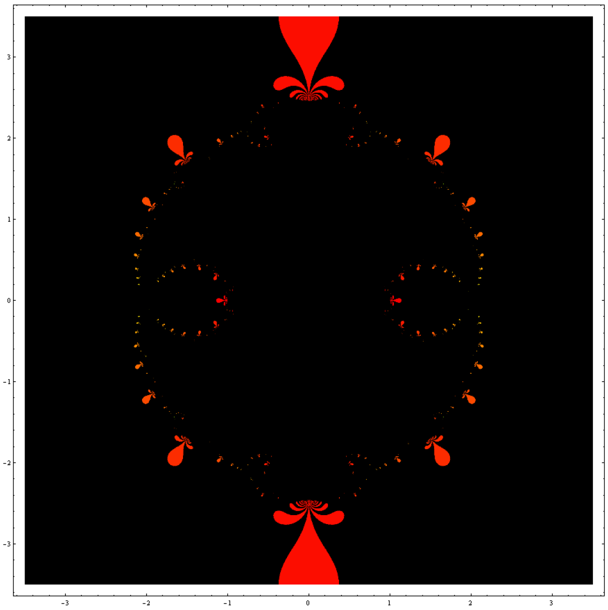

Figure 2: The Julia set is not connected

$$f(z) = \epsilon\{cz + d + \sum_{\infty}^{n=1} c_n(\frac{1}{(a_n - z)} - \frac{1}{a_n})\}, \qquad (1)$$

where, $\epsilon = \pm 1$, c, d, c_n, a_n are constants in $\mathbb{R}$, $0 \leq c < \infty$ and $\sum c_n/a^2 < \infty$.

The transcendental meromorphic functions represented by (1) have the poles and zeros in $\mathbb{R}$. $f(z)$ is a real meromorphic function and maps the upper half plane to itself. The following remark was given in [1].

Remark 2.1. If f is a transcendental meromorphic function of the form (1), then $J(f) = \mathbb{R}$ or $J(f)$ is a totally discontinuous subset of $\mathbb{R}$

With this remark we can give several examples for this kind of functions for which the Fatou set set can be simply or multiply connected, see examples in [1].

3 Conclusions

Taking some families in class $\mathcal{K} \cap S_k$ we give examples of the Fatou set which can be either simply-connected (Figure 1) or multiply-connected (Figure 2) in the last case the Julia set is a totally disconnected set.

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Competing interests

The authors declare that no competing interests exist.

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