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Existence and Regularity of Solutions to a System of Degenerate Nonlinear Elliptic Equations

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper, we shall study solvability and regularity properties of solutions to the system of equations:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^{(j)}(x, \nabla u_1, \nabla u_2) + g^{(j)}(x, u_1, u_2) = f^{(j)}(x) \ \text{ in } \Omega, \ j = 1, 2,$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n > 2$.

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1 Introduction

In this paper we prove the existence and regularity properties of solutions of Dirichlet problem for the following system of equations:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^{(j)}(x, \nabla u_1, \nabla u_2) + g^{(j)}(x, u_1, u_2) = f^{(j)}(x) \ \text{ in } \Omega, \ j = 1, 2, \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n > 2$. Moreover, n satisfies the following inequality: $1 < p_j < n$, $j = 1, 2$, where p_j is connected with the rate of growth of coefficients $a_i^{(j)}$ of the equations with respect to the first-order derivatives of unknown functions u_1, u_2.

The assumed conditions on coefficients $a_i^{(j)}$ and $g^{(j)}$ $(j = 1, 2)$ and known results of the theory of monotone operators allow us to prove existence of a generalized solution of our Dirichlet problem (see section 4). Then, we establish a theorem on boundedness of generalized solution of the problem (see section 6). Section 7 contained results on Hölder continuity of generalized solution of the same Dirichlet problem. The proof is based on the iterating Moser method, suitably modified and applied in the case of the equations (see, for instance [1], [2]). Finally, in conclusive section we consider an example fulfilling all our assumptions.

We note that, in non degenerate case, boundedness and regularity of generalized solution for one second order nonlinear elliptic equation were studied by many authors, see for instance [3] - [5] and for an elliptic system [6], [7]. Finally, concerning solvability and properties of solutions of nonlinear equations, in degenerate case, we refer, for instance, to [8, 9, 10] and [11] - [13].

2 Preliminaries

We shall suppose that $\mathbb{R}^n$ $(n > 2)$ is n-dimensional euclidian space with elements $x = (x_1, x_2, ..., x_n)$. Let Ω be a bounded open set of $\mathbb{R}^n$. Let p_j be a real number such that $1 < p_j < n$, $j = 1, 2$.

Hypothesis 2.1 Let $\nu_j : \Omega \to \mathbb{R}^+$ be a measurable function $(j = 1, 2)$ such that

$$\nu_j \in L^1_{loc}(\Omega) \ , \left(\frac{1}{\nu_j}\right)^{\frac{1}{p_j - 1}} \in L^1_{loc}(\Omega).$$

We denote by $W^{1,p_j}(\nu_j, \Omega)$ $(j = 1, 2)$ the set of all functions $u \in L^{p_j}(\Omega)$ having, for every $i = 1, ..., n$, the weak derivative $\frac{\partial u}{\partial x_i}$ with the property $\nu_j \left|\frac{\partial u}{\partial x_i}\right|^{p_j} \in L^1(\Omega)$. $W^{1,p_j}(\nu_j, \Omega)$ is a Banach space with respect to the norm

$$\|u\|_{1,p_j,\nu_j} = \left(\int_\Omega |u|^{p_j} + \sum_{i=1}^{n} \nu_j \left|\frac{\partial u}{\partial x_i}\right|^{p_j} dx\right)^{\frac{1}{p_j}}.$$

$\mathring{W}^{1,p_j}(\nu_j, \Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p_j}(\nu_j, \Omega)$.

We assume that there exists a number $t_j > \max\left(\frac{n}{p_j}, \frac{1}{p_j - 1}\right)$ $(j = 1, 2)$, such that

$$\frac{1}{\nu_j} \in L^{t_j}(\Omega).$$

For every $j = 1, 2$, we set $\tilde{p}_j = \dfrac{np_j}{n - p_j + n/t_j}$. Then, we have that $\mathring{W}^{1,p_j}(\nu_j, \Omega) \subset L^{\tilde{p}_j}(\Omega)$ and there exists $\hat{c}_j > 0$ depending only on n, p_j, t_j such that for every $u \in \mathring{W}^{1,p_j}(\nu_j, \Omega)$

$$(2.1)\quad \left(\int_\Omega |u|^{\tilde{p}_j} dx\right)^{1/\tilde{p}_j} \leq \hat{c}_j \left\{\int_{suppu} \left(\frac{1}{\nu_j}\right)^{t_j} dx\right\}^{1/p_j t_j} \left\{\int_\Omega \sum_{i=1}^n \nu_j \left|\frac{\partial u}{\partial x_i}\right|^{p_j} dx\right\}^{1/p_j}.$$

In this connection see for instance [14], [9] and [10].

3 Statement of the Problem

Let, for every $i = 1, ..., n,\ j = 1, 2,\ a_i^{(j)} : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, be Carathéodory functions. There exist $c_\alpha > 0,\ \alpha = 1, 2, ..., 9,\ \overline{p}_j \in (0, p_j),\ j = 1, 2$, such that, for almost every $x \in \Omega$ and every $\eta^{(1)}$, $\eta^{(2)} \in \mathbb{R}^n$ the next inequalities hold:

$$(3.1)\quad \sum_{i=1}^n a_i^{(1)}(x, \eta^{(1)}, \eta^{(2)})\eta_i^{(1)} \geq c_1\nu_1(x) \sum_{i=1}^n |\eta_i^{(1)}|^{p_1} - c_2[\nu_2(x)]^{\overline{p}_2/p_2} \sum_{i=1}^n |\eta_i^{(2)}|^{\overline{p}_2},$$

$$(3.2)\quad \sum_{i=1}^n a_i^{(2)}(x, \eta^{(1)}, \eta^{(2)})\eta_i^{(2)} \geq c_4\nu_2(x) \sum_{i=1}^n |\eta_i^{(2)}|^{p_2} - c_5[\nu_1(x)]^{\overline{p}_1/p_1} \sum_{i=1}^n |\eta_i^{(1)}|^{\overline{p}_1},$$

$$(3.3)\quad \begin{aligned} &\sum_{i=1}^n [\nu_1(x)]^{-1/(p_1-1)} |a_i^{(1)}(x, \eta^{(1)}, \eta^{(2)})|^{p_1/(p_1-1)} \leq \\ &\leq c_7 \left\{\nu_1(x) \sum_{i=1}^n |\eta_i^{(1)}|^{p_1} + [\nu_2(x)]^{\overline{p}_2/p_2} \sum_{i=1}^n |\eta_i^{(2)}|^{\overline{p}_2} + 1\right\}, \end{aligned}$$

$$(3.4)\quad \begin{aligned} &\sum_{i=1}^n [\nu_2(x)]^{-1/(p_2-1)} |a_i^{(2)}(x, \eta^{(1)}, \eta^{(2)})|^{p_2/(p_2-1)} \leq \\ &\leq c_8 \left\{\nu_2(x) \sum_{i=1}^n |\eta_i^{(2)}|^{p_2} + [\nu_1(x)]^{\overline{p}_1/p_1} \sum_{i=1}^n |\eta_i^{(1)}|^{\overline{p}_1} + 1\right\}. \end{aligned}$$

Moreover, we shall assume that for almost every $x \in \Omega$ and every $\eta^{(1)}$, $\eta^{(2)}$, $\overline{\eta}^{(1)}$, $\overline{\eta}^{(2)} \in \mathbb{R}^n$,

$$(3.5)\quad \begin{aligned} &\sum_{i=1}^n [a_i^{(1)}(x, \eta^{(1)}, \eta^{(2)}) - a_i^{(1)}(x, \overline{\eta}^{(1)}, \overline{\eta}^{(2)})](\eta_i^{(1)} - \overline{\eta}_i^{(1)}) + \\ &+ \sum_{i=1}^n [a_i^{(2)}(x, \eta^{(1)}, \eta^{(2)}) - a_i^{(2)}(x, \overline{\eta}^{(1)}, \overline{\eta}^{(2)})](\eta_i^{(2)} - \overline{\eta}_i^{(2)}) \geq 0. \end{aligned}$$

Let $\sigma_1 \in (0, p_1)$, $\sigma_2 \in (0, p_2)$. Let for every $j = 1, 2$, $g^{(j)} : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. We shall suppose that for almost every $x \in \Omega$ and every $u_1, u_2 \in \mathbb{R}$,

$$(3.6)\quad g^{(1)}(x, 0, 0) = 0,\ g^{(2)}(x, 0, 0) = 0,$$

(3.7) $\quad |g^{(1)}(x,u_1,u_2)|^{p_1/(p_1-1)} + |g^{(2)}(x,u_1,u_2)|^{p_2/(p_2-1)} \leq c_3(|u_1|^{p_1} + |u_2|^{p_2} + 1),$

(3.8) $\quad g^{(1)}(x,u_1,u_2)u_1 \geq -c_9|u_2|^{\sigma_2},$

(3.9) $\quad g^{(2)}(x,u_1,u_2)u_2 \geq -c_6|u_1|^{\sigma_1}.$

Finally, we shall assume that for almost every $x \in \Omega$ and every $u_1,\ u_2,\ u_1',\ u_2' \in \mathbb{R}$,

$$
\begin{aligned}
&[g^{(1)}(x,u_1,u_2) - g^{(1)}(x,u_1',u_2')](u_1 - u_1')+ \\
&\quad +[g^{(2)}(x,u_1,u_2) - g^{(2)}(x,u_1',u_2')](u_2 - u_2') \geq 0.
\end{aligned} \tag{3.10}
$$

Fix $f_j \in L^{p_j/(p_j-1)}(\Omega)$, $j = 1,2$.

Definition 3.1 We shall say that a pair (u_1,u_2) is a generalized solution of the Dirichlet problem for system (1.1), if $(u_1,u_2) \in \mathring{W}^{1,p_1}(\nu_1,\Omega) \times \mathring{W}^{1,p_2}(\nu_2,\Omega)$ and

$$\sum_{j=1,2} \int_\Omega \left\{ \sum_{i=1}^n a_i^{(j)}(x,\nabla u_1,\nabla u_2)\frac{\partial v_j}{\partial x_i} + g^{(j)}(x,u_1,u_2)v_j \right\} dx = \sum_{j=1,2} \int_\Omega f_j v_j dx$$

for every $(v_1,v_2) \in \mathring{W}^{1,p_1}(\nu_1,\Omega) \times \mathring{W}^{1,p_2}(\nu_2,\Omega)$.

4 Existence of Solutions

We shall prove the following

Theorem 4.1 *Under the above-stated assumptions on the function $a_i^{(j)}$, $g^{(j)}$ and f_j and Hypothesis 2.1 there exists a generalized solution of the Dirichlet problem for system (1.1).*

Proof: Define the operator $\mathcal{A} : \mathring{W}^{1,p_1}(\nu_1,\Omega) \times \mathring{W}^{1,p_2}(\nu_2,\Omega) \to (\mathring{W}^{1,p_1}(\nu_1,\Omega) \times \mathring{W}^{1,p_2}(\nu_2,\Omega))^\star$ by

$$\langle \mathcal{A}(u_1,u_2),(v_1,v_2)\rangle = \sum_{j=1,2} \int_\Omega \left\{ \sum_{i=1}^n a_i^{(j)}(x,\nabla u_1,\nabla u_2)\frac{\partial v_j}{\partial x_i} + g^{(j)}(x,u_1,u_2)v_j \right\} dx.$$

Due to (3.3), (3.4) and (3.7) we have that the operator $\mathcal{A}$ is well defined, bounded and, moreover, is demicontinuous. By (3.5) and (3.10) we have that $\mathcal{A}$ is monotone. From (3.6) and (3.10) it follows that for almost every $x \in \Omega$ and every $u_1, u_2 \in \mathbb{R}$,

$$g^{(1)}(x,u_1,u_2)u_1 + g^{(2)}(x,u_1,u_2)u_2 \geq 0.$$

Next, taking into account that for arbitrary fixed $(u_1,u_2) \in \mathring{W}^{1,p_1}(\nu_1,\Omega) \times \mathring{W}^{1,p_2}(\nu_2,\Omega)$, we have

$$\langle \mathcal{A}(u_1,u_2),(u_1,u_2)\rangle \geq \frac{c_1}{2}\|u_1\|_{1,p_1,\nu_1}^{p_1} + \frac{c_4}{2}\|u_2\|_{1,p_2,\nu_2}^{p_2} - c_{10}\text{meas}\Omega,$$

where positive constant c_{10} depends only on known parameters, then, we can conclude that the operator $\mathcal{A}$ is coercive.

Now, define the operator $\mathcal{F} : \mathring{W}^{1,p_1}(\nu_1,\Omega) \times \mathring{W}^{1,p_2}(\nu_2,\Omega) \to \mathbb{R}$ by

$$\langle \mathcal{F},(v_1,v_2)\rangle = \int_\Omega (f_1 v_1 + f_2 v_2)dx.$$

We have that $\mathcal{F} \in (\mathring{W}^{1,p_1}(\nu_1,\Omega) \times \mathring{W}^{1,p_2}(\nu_2,\Omega))^\star$.

Then, from well-known results of the theory of monotone operators (see for instance [15]), there exists $(u_1,u_2) \in \mathring{W}^{1,p_1}(\nu_1,\Omega) \times \mathring{W}^{1,p_2}(\nu_2,\Omega)$ such that $\mathcal{A}(u_1,u_2) = \mathcal{F}$. Therefore, the pair (u_1,u_2) is a generalized solution of the Dirichlet problem for system (1.1).

5 Auxiliary Results

Let $h \in C^\infty(\mathbb{R})$ be a non-decreasing function, such that $h = 0$ on $]-\infty, 0]$ and $h = 1$ on $[1, +\infty[$.

We set

$$\tilde{c}_1 = \max_{\mathbb{R}} |h'|.$$

Let for every $s \in \mathbb{N}$, $h_s : \mathbb{R} \to \mathbb{R}$ be the function such that

$$h_s(\eta) = \eta + (s + 1 - \eta)h(\eta - s) - (s + 1 + \eta)h(-\eta - s), \qquad \eta \in \mathbb{R}.$$

We have $\{h_s\} \subseteq C^\infty(\mathbb{R})$, and for every $s \in \mathbb{N}$ the following property hold:

$$h_s(\eta) = \eta \quad if \quad |\eta| \le s$$

$$h_s(\eta) = -s - 1 \quad if \quad \eta \le -s - 1$$

$$h_s(\eta) = s + 1 \quad if \quad \eta \ge s + 1.$$

Moreover, for every $s \in \mathbb{N}$ and $\eta \in \mathbb{R}$, we have

$$|h_s(\eta)| \le 2|\eta|,$$

$$0 \le h_s'(\eta) \le \tilde{c_1}, \qquad |\eta| h_s'(\eta) \le 2\tilde{c_1}|h_s(\eta)|. \tag{5.1}$$

By d_i, $i = 1, 2, ...$, we shall denote positive constants which depend only on $n, p_j, \overline{p}_j, c_1, c_2, ..., c_9$, $\tilde{c}_1, \hat{c}_j$ and on measΩ $(j = 1, 2)$.

6 Boundedness of Solutions

To achieve boundedness of generalized solution of (1.1) we need the following

Hypothesis 6.1 Let the following conditions be satisfied:

$$|f_j|^{p_j/(p_j-1)} \in L^{\tau_j}(\Omega) \text{ with } \tau_j > \frac{nt_j}{p_j t_j - n} \ (j = 1, 2),$$

$$\overline{p}_1 < p_1 \left(\frac{p_2}{n} - \frac{1}{t_2} \right), \qquad \overline{p}_2 < p_2 \left(\frac{p_1}{n} - \frac{1}{t_1} \right),$$

$$\sigma_2 < \left(\frac{p_1}{n} - \frac{1}{t_1} \right) \frac{np_2 t_2}{n(1 + t_2) - p_2 t_2}.$$

We shall prove the following

Theorem 6.2 *Let Hypotheses 2.1 and 6.1 hold. Let a pair (u_1, u_2) be a generalized solution of the Dirichlet problem for system (1.1). Then u_1, $u_2 \in L^\infty(\Omega)$.*

Proof:

Define

$$\varphi_1 = 1 + |f_1|^{p_1/(p_1-1)} + |u_2|^{\sigma_2} + \nu_2^{\overline{p}_2/p_2} \sum_{i=1}^n \left| \frac{\partial u_2}{\partial x_i} \right|^{\overline{p}_2},$$

$$t = \min \left(\tau_1, \frac{p_2}{\overline{p}_2}, \frac{\tilde{p}_2}{\sigma_2} \right).$$

We have that $\varphi_1 \in L^t(\Omega)$ and $t > nt_1/(p_1t_1 - n)$.

Let $s \in \mathbb{N}$ and $r > 0$. Set

$$v_1 = u_1[1 + h_s^2(u_1)]^r,$$

$$\omega_1 = [1 + h_s^2(u_1)]^r + 2r[1 + h_s^2(u_1)]^{r-1}h_s(u_1)h_s'(u_1)u_1.$$

By (5.1) the function $v_1 \in \mathring{W}^{1,p_1}(\nu_1, \Omega)$ and, for every $i = 1, 2, ...n$,

$$\frac{\partial v_1}{\partial x_i} = \omega_1 \frac{\partial u_1}{\partial x_i} \text{ a.e. in } \Omega.$$

Choosing $(v_1, 0)$ as test function, we have

$$\int_\Omega \left\{ \sum_{i=1}^n a_i^{(1)}(x, \nabla u_1, \nabla u_2) \frac{\partial u_1}{\partial x_i} \omega_1 + g^{(1)}(x, u_1, u_2)v_1 \right\} dx = \int_\Omega f_1 v_1 dx.$$

Note that $[1 + h_s^2(u_1)]^r \leq \omega_1 \leq (1 + 4\tilde{c}_1 r)[1 + h_s^2(u_1)]^r$ in Ω, then due to (3.1)

$$c_1 \int_\Omega \nu_1 \sum_{i=1}^n \left| \frac{\partial u_1}{\partial x_i} \right|^{p_1} [1 + h_s^2(u_1)]^r dx + \int_\Omega g^{(1)}(x, u_1, u_2)v_1 dx \leq$$

$$\leq c_2(1 + 4\tilde{c}_1 r) \int_\Omega \nu_2^{\overline{p}_2/p_2} \sum_{i=1}^n \left| \frac{\partial u_2}{\partial x_i} \right|^{\overline{p}_2} [1 + h_s^2(u_1)]^r dx + \int_\Omega f_1 v_1 dx. \tag{6.1}$$

From (3.8) it follows that

$$-c_9 \int_\Omega |u_2|^{\sigma_2}[1 + h_s^2(u_1)]^r dx \leq \int_\Omega g^{(1)}(x, u_1, u_2)v_1 dx \tag{6.2}$$

and from the definition of the function v_1 and Young inequality we obtain that

$$\int_\Omega f_1 v_1 dx \leq \int_\Omega [|f_1|^{p_1/(p_1-1)} + |u_1|^{p_1}][1 + h_s^2(u_1)]^r dx. \tag{6.3}$$

From (6.1)-(6.3) it follows that

$$\int_\Omega \nu_1 \sum_{i=1}^n \left| \frac{\partial u_1}{\partial x_i} \right|^{p_1} [1 + h_s^2(u_1)]^r dx \leq d_1(1 + r) \int_\Omega [\varphi_1 - 1 + |u_1|^{p_1}][1 + h_s^2(u_1)]^r dx.$$

Last inequality implies that

$$\int_\Omega \nu_1 \sum_{i=1}^n \left| \frac{\partial u_1}{\partial x_i} \right|^{p_1} [1 + h_s^2(u_1)]^r dx \leq$$

$$\leq d_1(1 + r) \int_\Omega [\varphi_1 + |u_1|^{a_1}][1 + h_s^2(u_1)]^r dx \tag{6.4}$$

for every $r > 0$ and $a_1 > p_1$.

$\forall\, s \in \mathbb{N}$, $r > 0$, let

$$T_s(r) = 1 + \int_\Omega [\varphi_1 + |u_1|^{a_1}][1 + h_s^2(u_1)]^r dx.$$

It results

$$T_s(r) \le 1 + \int_\Omega |\tilde{u}_1|^{a_1}\, dx + ||\varphi_1||_{L^t(\Omega)} \left(\int_\Omega (\tilde{w}_1 + 1)^{\tilde{p}_1} dx \right)^{(t-1)/t},$$

where

$$\tilde{u}_1 = u_1[1 + h_s^2(u_1)]^{r/a_1},$$

$$\tilde{w}_1 = [1 + h_s^2(u_1)]^{rt/[\tilde{p}_1(t-1)]} - 1.$$

Now, if we take a_1 such that $p_1 < a_1 < \tilde{p}_1$, from last inequality we have

$$T_s(r) \le 1 + \left(\int_\Omega |\tilde{u}_1|^{\tilde{p}_1}\, dx \right)^{a_1/\tilde{p}_1} (\text{meas}\Omega)^{(\tilde{p}_1 - a_1)/\tilde{p}_1} +$$

$$+ \|\varphi_1\|_{L^t(\Omega)} 2^{\tilde{p}_1(t-1)/t} \left(\int_\Omega \tilde{w}_1^{\tilde{p}_1} dx \right)^{(t-1)/t} + \|\varphi_1\|_{L^t(\Omega)} 2^{\tilde{p}_1(t-1)/t} (\text{meas}\Omega)^{(t-1)/t}.$$

From (2.1) last inequality gives

$$T_s(r) \le d_2 + d_3 \left(\int_\Omega \nu_1 \sum_{i=1}^n \left| \frac{\partial \tilde{u}_1}{\partial x_i} \right|^{p_1} dx \right)^{a_1/p_1} +$$

(6.5)

$$+ d_4 \left(\int_\Omega \nu_1 \sum_{i=1}^n \left| \frac{\partial \tilde{w}_1}{\partial x_i} \right|^{p_1} dx \right)^{\tilde{p}_1(t-1)/p_1 t}.$$

For every $i = 1, 2, ..., n$, easy computations imply:

$$\left| \frac{\partial \tilde{u}_1}{\partial x_i} \right| \le d_5(r+1)[1 + h_s^2(u_1)]^{r/a_1} \left| \frac{\partial u_1}{\partial x_i} \right|, \tag{6.6}$$

$$\left| \frac{\partial \tilde{w}_1}{\partial x_i} \right| \le d_6 r [1 + h_s^2(u_1)]^{rt/[\tilde{p}_1(t-1)]} \left| \frac{\partial u_1}{\partial x_i} \right|. \tag{6.7}$$

From (6.5)-(6.7)

$$T_s(r) \le d_2 + d_7(r+1)^{a_1} \left(\int_\Omega \nu_1 \sum_{i=1}^n \left| \frac{\partial u_1}{\partial x_i} \right|^{p_1} [1 + h_s^2(u_1)]^{(rp_1)/a_1} dx \right)^{a_1/p_1} +$$

(6.8)

$$+ d_8 r^{[\tilde{p}_1(t-1)]/t} \left(\int_\Omega \nu_1 \sum_{i=1}^n \left| \frac{\partial u_1}{\partial x_i} \right|^{p_1} [1 + h_s^2(u_1)]^{(rp_1 t)/[\tilde{p}_1(t-1)]} dx \right)^{\tilde{p}_1(t-1)/p_1 t}.$$

We set

$$\Theta = \min\left(\frac{a_1}{p_1}, \frac{\tilde{p}_1(t-1)}{p_1 t} \right) \ (\Theta > 1).$$

From Hölder inequality and (6.8), $\forall\, s \in \mathbb{N},\ r > 0,$ we obtain:

$$T_s(r) \le d_2 + d_9(r+1)^{2a_1} \left(\int_\Omega \nu_1 \sum_{i=1}^n \left| \frac{\partial u_1}{\partial x_i} \right|^{p_1} [1 + h_s^2(u_1)]^{r/\Theta} dx \right)^{\Theta},$$

where the positive constant d_9 depends on known parameters and the $\|u_1\|_{1,p_1,\nu_1}$.

Choosing $r = r/\Theta$ in (6.4), from last inequality we have

$$T_s(r) \le d_{10}(r+1)^{3a_1}[T_s(r/\Theta)]^{\Theta},\ \forall r > 0. \tag{6.9}$$

Let us introduce a sequence $\{\rho_j\}$ such that

$$\rho_j = b\Theta^{j+1} \qquad\qquad \forall\, j \in \mathbb{N}_0,$$

where $b = \dfrac{1}{2\Theta} \min\left(\tilde{p}_1 - a_1, \dfrac{\tilde{p}_1(t-1)}{t}\right)$.

We have $\dfrac{\rho_j}{\Theta} = \rho_{j-1}$ and so, due to such and (6.9)

$$T_s(\rho_j) \le d_{10}(1+\rho_j)^{3a_1}[T_s(\rho_{j-1})]^{\Theta}.$$

Recursion relation and the inequality

$$T_s(\rho_0) \le d_{11} + d_{12}\int_\Omega |u_1|^{\tilde{p}_1}\,dx$$

allow to conclude that $\forall\, j = 1, 2, \ldots,$

$$T_s(\rho_j) \le d_{13}^{\Theta^j}$$

where d_{13} depends on known parameters, $||\varphi_1||_{L^t(\Omega)}$ and the $\|u_1\|_{1,p_1,\nu_1}$.

Now, noting that $h_s(u) \to u$ as $s \to \infty$, from definition of $T_s(r)$ and Fatou's lemma, it follows

$$\int_\Omega |u_1|^{a_1+\rho_j}\,dx \le (d_{13}^{\frac{1}{b\Theta}} + 1)^{a_1+\rho_j} \qquad\qquad j = 1, 2, \ldots,$$

and so $u_1 \in L^\infty(\Omega)$.

Analogous property we establish for the function $u_2(x)$, using considerations which are similar to the above given.

Theorem is so proved.

7 Regularity of Solutions

We shall prove the Hölder property of generalized solutions of (1.1) estimating Hölder's constant for an interior region of domain Ω.

We need the following

Hypothesis 7.1 For every $j = 1, 2,$ there exists a number $\bar{t}_j > \frac{nt_j}{p_jt_j-n}$ such that $[\nu_j(x)]^{\bar{t}_j} \in A_{1+\mu_j}$ (Muckenhoupt's class) with $0 < \mu_j < n/(p_jt_j - n)$.

For every $y \in \mathbb{R}^n$ and $\rho > 0$ we denote

$$B(y, \rho) = \{x \in \mathbb{R}^n :\ |x - y| < \rho\}.$$

Remark 7.2 Due to Hypothesis 7.1, there exists a positive constant c such that for every $j = 1, 2,$ every $y \in \Omega$ and $\rho > 0$ with $\overline{B(y, \rho)} \subset \Omega$ the following inequality hold:

$$\left\{\rho^{-n}\int_{B(y,\rho)}\left(\frac{1}{\nu_j}\right)^{t_j}\right\}^{1/t_j}\left\{\rho^{-n}\int_{B(y,\rho)}\nu_j^{\bar{t}_j}\,dx\right\}^{1/\bar{t}_j} \le c.$$

Remark 7.3 It may be useful to note that if $\nu_j(x) \in A_{1+\mu_j}$, then $[\nu_j(x)]^\tau \in A_{1+\mu_j}$ for some $\tau > 1$. So, if $\tau > nt_j/(p_jt_j - n)$ it will be sufficient to assume $\nu_j(x) \in A_{1+\mu_j}$.

Now, we can formulate the main result of this section:

Theorem 7.4 *Let the assumptions of Theorem 6.2 and Hypothesis 7.1 be satisfied. Let a pair* (u_1, u_2) *be a generalized solution of the Dirichlet problem for system* (1.1). *Then there exist positive constants* C', C'' *and* σ', σ'' (< 1) *such that for every open set* Ω', $\overline{\Omega'} \subset \Omega$ *and every* $x, y \in \Omega'$,

$$|u_1(x) - u_1(y)| \leq C'[d(\Omega', \Omega)]^{-\sigma'}|x - y|^{\sigma'},$$

$$|u_2(x) - u_2(y)| \leq C''[d(\Omega', \Omega)]^{-\sigma''}|x - y|^{\sigma''}.$$

proof.

By virtue of Theorem 6.2 we have $u_1, u_2 \in L^\infty(\Omega)$. Denote

$$M_i = ||u_i||_{L^\infty(\Omega)}, \; i = 1, 2,$$

$$a_1 = \frac{1}{p_1}\left(p_1 - \frac{n}{t_\star} - \frac{n}{t_1}\right), \text{ where } t_\star = \min\left\{\tau_1, \bar{t}_1, \frac{p_2}{\overline{p}_2}\right\}.$$

Let's fix $y \in \Omega$, $\rho > 0$ and $\overline{B(y, 2\rho)} \subset \Omega$. Let's put:

$$\omega_1 = \operatorname{ess}\inf_{B(y,2\rho)} u_1, \quad \omega_2 = \operatorname{ess}\sup_{B(y,2\rho)} u_1, \quad \omega = \omega_2 - \omega_1$$

We shall show that:

$$(7.1) \qquad osc\{u_1, B(y, \rho)\} \leq \chi_1\omega + \rho^{a_1},$$

with $\chi_1 \in]0, 1[$.

Obviously we will assume that

$$\omega \geq \rho^{a_1} \quad \text{(otherwise it is clear that (7.1) is true).}$$

Let $G_1 : \Omega \to \mathbb{R}$ be the functions such that

$$G_1 = \frac{2e\omega}{u_1 - \omega_1 + \rho^{a_1}}, \text{ in } B(y, 2\rho), \; G_1 = e, \text{ in } \Omega \setminus B(y, 2\rho).$$

It results $G_1 \geq e$ in Ω.

Let $\varphi \in C_0^\infty(\Omega)$: $0 \leq \varphi \leq 1$ in Ω, $\varphi = 0$ in $\Omega \setminus B(y, 2\rho)$ and

$$|\nabla\varphi| \leq \frac{\bar{c}}{\rho} \quad \text{in} \quad \Omega.$$

Let us fix $r \geq 0$ and $s > p_1$. We set

$$v_1 = (\lg G_1)^r G_1^{p_1-1}\varphi^s,$$

$$z_1 = -\frac{1}{2e\omega}[r(\lg G_1)^{r-1} + (p_1 - 1)(\lg G_1)^r]G_1^{p_1}\varphi^s.$$

We note that $v_1 \in \mathring{W}^{1,p_1}(\nu_1, \Omega)$ and, $\forall i = 1, 2, ..., n$, the next inequality is true:

$$(7.2) \quad \left|\frac{\partial v_1}{\partial x_i} - z_1\frac{\partial u_1}{\partial x_i}\right| \leq \bar{c}s\rho^{-1}(\lg G_1)^r G_1^{p_1-1}\varphi^{s-1} \text{ a.e. in } B(y, 2\rho).$$

Since (u_1, u_2) is a generalized solution of (1.1), choosing $(v_1, 0)$ as test function we obtain

$$\int_\Omega \left\{\sum_{i=1}^n a_i^{(1)}(x, \nabla u_1, \nabla u_2)\frac{\partial v_1}{\partial x_i} + g^{(1)}(x, u_1, u_2)v_1\right\} dx = \int_\Omega f_1 v_1 dx.$$

Hence

$$\int_\Omega \sum_{i=1}^n \left\{ a_i^{(1)}(x, \nabla u_1, \nabla u_2) \frac{\partial u_1}{\partial x_i} \right\} (-z_1) dx \le \int_\Omega [|f_1| + |g^{(1)}(x, u_1, u_2)|] v_1 dx +$$

$$+ \int_\Omega \sum_{i=1}^n |a_i^{(1)}(x, \nabla u_1, \nabla u_2)| \left| \frac{\partial v_1}{\partial x_i} - z_1 \frac{\partial u_1}{\partial x_i} \right| dx.$$

Taking into account that

$$(2e\omega)^{-1}(p_1 - 1)(\lg G_1)^r G_1^{p_1} \varphi^s \le -z_1 \le (2e\omega)^{-1} p_1 (1 + r)(\lg G_1)^r G_1^{p_1} \varphi^s,$$

from (3.1), last inequality implies

$$\frac{(p_1 - 1)c_1}{2e\omega} \int_\Omega \nu_1 \sum_{i=1}^n \left| \frac{\partial u_1}{\partial x_i} \right|^{p_1} (\lg G_1)^r G_1^{p_1} \varphi^s dx \le \frac{c_2 p_1 (1 + r)}{2e\omega} \cdot$$

(7.3)

$$\cdot \int_\Omega \nu_2^{\overline{p}_2/p_2} \sum_{i=1}^n \left| \frac{\partial u_2}{\partial x_i} \right|^{\overline{p}_2} (\lg G_1)^r G_1^{p_1} \varphi^s dx + \int_\Omega [|f_1| + |g^{(1)}(x, u_1, u_2)|] v_1 dx + I,$$

where

$$I = \int_\Omega \sum_{i=1}^n |a_i^{(1)}(x, \nabla u_1, \nabla u_2)| \left| \frac{\partial v_1}{\partial x_i} - z_1 \frac{\partial u_1}{\partial x_i} \right| dx.$$

Now we shall obtain estimates for the addends in the right-hand side of inequality (7.3).

Using definition of G_1, we get

$$\int_\Omega \nu_2^{\overline{p}_2/p_2} \sum_{i=1}^n \left| \frac{\partial u_2}{\partial x_i} \right|^{\overline{p}_2} (\lg G_1)^r G_1^{p_1} \varphi^s dx \le$$

(7.4)

$$\le (2e\omega)^{p_1} \int_\Omega \nu_2^{\overline{p}_2/p_2} \sum_{i=1}^n \left| \frac{\partial u_2}{\partial x_i} \right|^{\overline{p}_2} (\lg G_1)^r \varphi^s \rho^{-a_1 p_1} dx.$$

Moreover, due to (3.7)

$$\int_\Omega [|f_1| + |g^{(1)}(x, u_1, u_2)|] v_1 dx \le$$

(7.5)

$$\le (c_3 + 1)(2e\omega)^{p_1 - 1}(M_1^{p_1} + M_2^{p_2} + 1)(\mathrm{diam}\Omega + 1) \int_\Omega (1 + |f_1|)(\lg G_1)^r \varphi^s \rho^{-a_1 p_1} dx.$$

Inequalities (7.3)-(7.5) imply

$$\frac{(p_1 - 1)c_1}{2e\omega} \int_{B(y,2\rho)} \nu_1 \sum_{i=1}^n \left| \frac{\partial u_1}{\partial x_i} \right|^{p_1} (\lg G_1)^r G_1^{p_1} \varphi^s dx \le \beta_1 (2e\omega)^{p_1 - 1}(1 + r) \cdot$$

(7.6)

$$\cdot \int_{B(y,2\rho)} \rho^{-a_1 p_1} \left\{ 1 + |f_1| + \nu_2(x)^{\overline{p}_2/p_2} \sum_{i=1}^n \left| \frac{\partial u_2}{\partial x_i} \right|^{\overline{p}_2} \right\} (\lg G_1)^r \varphi^s dx + I,$$

where $\beta_1 > 0$ depends only on $p_1, p_2, c_2, c_3, M_1, M_2$ and diam Ω.

Let us estimate I. Due to (7.2) we have

(7.7) $|a_i^{(1)}(x, \nabla u_1, \nabla u_2)| \left|\frac{\partial v_1}{\partial x_i} - z_1 \frac{\partial u_1}{\partial x_i}\right| \leq |a_i^{(1)}(x, \nabla u_1, \nabla u_2)| \frac{\bar{c}s}{\rho} (\lg G_1)^r G_1^{p_1 - 1} \varphi^{s-1}$

a.e. in Ω.

Let $\epsilon > 0$. Then, with the use of Young inequality for every $x \in \Omega : \varphi(x) \neq 0$ we obtain

$$|a_i^{(1)}(x, \nabla u_1, \nabla u_2)| \bar{c} s \rho^{-1} (\lg G_1)^r G_1^{p_1-1} \varphi^{s-1} =$$
$$= \left\{ \epsilon^{(p_1-1)/p_1} \nu_1^{-1/p_1} |a_i^{(1)}(x, \nabla u_1, \nabla u_2)| \frac{G_1^{p_1-1}}{(2e\omega)^{p_1-1}} \epsilon^{-(p_1-1)/p_1} \nu_1^{1/p_1} \bar{c} s \frac{1}{\rho} \varphi^{-1} \right\} \cdot$$
$$\cdot (2e\omega)^{p_1-1} (\lg G_1)^r \varphi^s \leq$$

(7.8)

$$\leq \left\{ \epsilon \nu_1^{-1/(p_1-1)} |a_i^{(1)}(x, \nabla u_1, \nabla u_2)|^{p_1/(p_1-1)} \frac{G_1^{p_1}}{(2e\omega)^{p_1}} + \epsilon^{1-p_1} \nu_1 (\bar{c}s)^{p_1} \rho^{-p_1} \varphi^{-p_1} \right\} \cdot$$
$$\cdot (2e\omega)^{p_1-1} (\lg G_1)^r \varphi^s.$$

From (7.7) and (7.8) we have

$$I \leq \frac{\epsilon}{2e\omega} \int_\Omega \nu_1^{-1/(p_1-1)} \sum_{i=1}^n |a_i^{(1)}(x, \nabla u_1, \nabla u_2)|^{p_1/(p_1-1)} G_1^{p_1} (\lg G_1)^r \varphi^s dx +$$

(7.9)

$$+ \epsilon^{1-p_1} (2e\omega)^{p_1-1} n (\bar{c}s)^{p_1} \int_\Omega \rho^{-p_1} \nu_1(x) (\lg G_1)^r \varphi^{s-p_1} dx.$$

On the other hand, (3.3) implies

$$\sum_{i=1}^n \nu_1^{-1/(p_1-1)} |a_i^{(1)}(x, \nabla u_1, \nabla u_2)|^{p_1/(p_1-1)} \leq$$

(7.10)

$$\leq c_7 \left\{ \sum_{i=1}^n \nu_1 \left|\frac{\partial u_1}{\partial x_i}\right|^{p_1} + [\nu_2]^{\overline{p}_2/p_2} \sum_{i=1}^n \left|\frac{\partial u_2}{\partial x_i}\right|^{\overline{p}_2} + 1 \right\}$$

a.e. in Ω.

Then, using (7.9) and (7.10) we obtain

$$I \leq \frac{\epsilon c_7}{2e\omega} \int_{B(y,2\rho)} \nu_1 \sum_{i=1}^n \left|\frac{\partial u_1}{\partial x_i}\right|^{p_1} G_1^{p_1} (\lg G_1)^r \varphi^s dx +$$

(7.11)

$$+ (2e\omega)^{p_1-1} \beta_2 s^{p_1} (\epsilon + \epsilon^{1-p_1}) \int_{B(y,2\rho)} \Psi_1 (\lg G_1)^r \varphi^{s-p_1} dx,$$

where

$$\Psi_1 = \rho^{-p_1}\nu_1 + \rho^{-a_1 p_1}\left\{1 + |f_1| + \nu_2^{\overline{p}_2/p_2}\sum_{i=1}^{n}\left|\frac{\partial u_2}{\partial x_i}\right|^{\overline{p}_2}\right\},$$

and the positive constant β_2 depends only on n, p_1, c_7 and diam Ω.

Due to Hypotheses 6.1, 7.1 it is convenient to observe that $\Psi_1 \in L^{t_\star}(\Omega)$ and $t_\star > \frac{nt_1}{p_1 t_1 - n}$.

From (7.6) and (7.11), we establish that

$$\frac{(p_1-1)c_1}{2e\omega}\int_{B(y,2\rho)}\nu_1\sum_{i=1}^{n}\left|\frac{\partial u_1}{\partial x_i}\right|^{p_1}(\lg G_1)^r G_1^{p_1}\varphi^s dx \le$$

$$\le \beta_1(2e\omega)^{p_1-1}(1+r)\int_{B(y,2\rho)}\Psi_1(\lg G_1)^r\varphi^{s-p_1}dx +$$

$$+\frac{\epsilon c_7}{2e\omega}\int_{B(y,2\rho)}\nu_1\sum_{i=1}^{n}\left|\frac{\partial u_1}{\partial x_i}\right|^{p_1}(\lg G_1)^r G_1^{p_1}\varphi^s dx +$$

$$+(2e\omega)^{p_1-1}\beta_2 s^{p_1}(\epsilon+\epsilon^{1-p_1})\int_{B(y,2\rho)}\Psi_1(\lg G_1)^r\varphi^{s-p_1}dx.$$

Setting $\epsilon = \dfrac{(p_1-1)c_1}{2c_7}$, from the last inequality we get

$$\int_{B(y,2\rho)}\nu_1\sum_{i=1}^{n}\left|\frac{\partial u_1}{\partial x_i}\right|^{p_1}(\lg G_1)^r G_1^{p_1}\varphi^s dx \le$$

(7.12)

$$\le (2e\omega)^{p_1}\beta_3 s^{p_1}(r+1)\int_{B(y,2\rho)}\Psi_1(\lg G_1)^r\varphi^{s-p_1}dx,$$

where the positive constant β_3 depends only on $n, p_1, p_2, c_1, c_2, c_7, \mathrm{diam}\Omega$, and M_1, M_2.

Now, if we assume that $\varphi = 1$ in $B(y, \dfrac{3}{2}\rho)$, from (7.12), with $r = 0$ and $s = p_1 + 1$, we have

$$\int_{B(y,\frac{3}{2}\rho)}\nu_1\sum_{i=1}^{n}\left|\frac{\partial u_1}{\partial x_i}\right|^{p_1}G_1^{p_1}dx \le (2e\omega)^{p_1}\beta_3(p_1+1)^{p_1}\int_{B(y,2\rho)}\Psi_1 dx. \tag{7.13}$$

If we take in (7.12) instead of φ the function $\varphi_1 \in C_0^\infty(\Omega)$ such that: $0 \le \varphi_1 \le 1$ in Ω, $\varphi_1 = 1$ in $B(y,\rho)$, $\varphi_1 = 0$ in $\Omega \setminus B(y, \frac{3}{2}\rho)$ and $|\nabla\varphi_1| \le \frac{\overline{c}}{\rho}$ in Ω, we obtain for every $r > 0$ and $s > p_1$,

$$\int_{B(y,2\rho)}\nu_1\sum_{i=1}^{n}\left|\frac{\partial u_1}{\partial x_i}\right|^{p_1}(\lg G_1)^r G_1^{p_1}\varphi_1^s dx \le$$

(7.14)

$$\le (2e\omega)^{p_1}\beta_3 s^{p_1}(r+1)\int_{B(y,2\rho)}\Psi_1(\lg G_1)^r\varphi_1^{s-p_1}dx.$$

We set

$$\theta = \frac{\tilde{p}_1(t_\star-1)}{p_1 t_\star}, \quad m = \frac{p_1 t_\star}{t_\star - 1},$$

and for every $r, s > 0$ we define

$$I(r,s) = \int_{B(y,2\rho)}(\lg G_1)^r\varphi_1^s dx.$$

We define $z = (\lg G_1)^{r/\tilde{p}_1}\varphi_1^{s/\tilde{p}_1}$.

We observe that $z \in \mathring{W}^{1,p_1}(\nu_1,\Omega)$ and, for every $i = 1, 2, ..., n$,

$$\int_\Omega \nu_1 \left|\frac{\partial z}{\partial x_i}\right|^{p_1} dx \le \left(\frac{r}{2e\omega}\right)^{p_1} \int_{B(y,2\rho)} \nu_1 \sum_{i=1}^n \left|\frac{\partial u_1}{\partial x_i}\right|^{p_1} (\lg G_1)^{rp_1/\tilde{p}_1} G_1^{p_1} \varphi_1^{sp_1/\tilde{p}_1} dx +$$

(7.15)

$$+(\bar{c}s)^{p_1} \int_{B(y,2\rho)} \rho^{-p_1} \nu_1 (\lg G_1)^{rp_1/\tilde{p}_1} \varphi_1^{[(s/\tilde{p}_1)-1]p_1} dx.$$

Using (7.14), (7.15) and taking into account that $\Psi_1 \in L^{t_\star}(\Omega)$, we obtain that for every $i = 1, 2, ..., n$,

$$\int_\Omega \nu_1 \left|\frac{\partial z}{\partial x_i}\right|^{p_1} dx \le \beta_4 s^{p_1} (r+1)^{p_1+1} \left(\int_{B(y,2\rho)} \Psi_1^{t_\star} dx\right)^{1/t_\star} \left[I\left(\frac{r}{\theta}, \frac{s}{\theta} - m\right)\right]^{(t_\star - 1)/t_\star},$$

where the positive constant β_4 depends only on $n, p_1, t_1, c_1, c_2, c_7, \bar{c}, \text{diam}\Omega$, and M_1, M_2.

Last inequality, definition of $I(r, s)$ and (2.1) give

$$I(r, s) \le \beta_5 s^{\tilde{p}_1} (1 + r)^{(\tilde{p}_1/p_1)(p_1+1)}.$$

(7.16)

$$\cdot \left\{\left[\int_{B(y,2\rho)} \left(\frac{1}{\nu_1}\right)^{t_1} dx\right]^{1/t_1} \left[\int_{B(y,2\rho)} \Psi_1^{t_\star} dx\right]^{1/t_\star}\right\}^{\tilde{p}_1/p_1} \left[I\left(\frac{r}{\theta}, \frac{s}{\theta} - m\right)\right]^\theta,$$

where the positive constant β_5 depends only on $n, p_1, t_1, c_1, c_2, c_7, \hat{c_1}, \bar{c}, \text{diam}\Omega$, and M_1, M_2.

We denote by $\gamma(u_2, f_1)$ the norm of function

$$1 + |f_1| + \nu_2^{\overline{p}_2/p_2} \sum_{i=1}^n \left|\frac{\partial u_2}{\partial x_i}\right|^{\overline{p}_2}$$

in $L^{t_\star}(\Omega)$. Then, we have

$$\left(\int_{B(y,2\rho)} \Psi_1^{t_\star} dx\right)^{1/t_\star} \le \rho^{-p_1} \left(\int_{B(y,2\rho)} \nu_1^{t_\star} dx\right)^{1/t_\star} + \rho^{-a_1 p_1} \gamma(u_2, f_1). \tag{7.17}$$

Using (7.17) and Remark 7.2, we obtain

$$\left[\int_{B(y,2\rho)} \left(\frac{1}{\nu_1}\right)^{t_1} dx\right]^{1/t_1} \left[\int_{B(y,2\rho)} \Psi_1^{t_\star} dx\right]^{1/t_\star} \le$$

$$\le c2^{(n/t_1 + n/t_\star)} (\chi_n + 1) \rho^{(-p_1 + n/t_\star + n/t_1)} + \rho^{-a_1 p_1} \gamma(u_2, f_1) ||1/\nu_1||_{L^{t_1}(\Omega)},$$

where χ_n is the measure of the unit ball in $\mathbb{R}^n$.

Due to the definition of a_1, last inequality implies

$$\left[\int_{B(y,2\rho)} \left(\frac{1}{\nu_1}\right)^{t_1} dx\right]^{1/t_1} \left[\int_{B(y,2\rho)} \Psi_1^{t_\star} dx\right]^{1/t_\star} \le \beta_6 \rho^{(-p_1 + n/t_\star + n/t_1)}, \tag{7.18}$$

where the positive constant β_6 depends only on $n, p_1, t_1, t_\star, \gamma(u_2, f_1), ||1/\nu_1||_{L^{t_1}(\Omega)}$ and χ_n.

Note that due to definition of $\tilde{p}_1$ and θ we have

$$(7.19) \qquad \left(p_1 - \frac{n}{t_\star} - \frac{n}{t_1}\right)\frac{\tilde{p}_1}{p_1} = n(\theta - 1).$$

From (7.16), (7.18) and (7.19) we get

$$(7.20) \quad I(r,s) \leq \beta_7 (r+s)^{m_1} \rho^{-n(\theta-1)} \left[I\left(\frac{r}{\theta}, \frac{s}{\theta} - m\right)\right]^\theta,$$

where

$$\beta_7 = \beta_5 \beta_6^{\tilde{p}_1/p_1}, \ m_1 = 2\tilde{p}_1 \frac{(p_1+1)}{p_1}.$$

Now we set for $j = 0, 1, 2, ...$

$$r_j = \frac{t_1 p_1}{t_1 + 1}\theta^j, \ s_j = \frac{m\theta}{\theta - 1}(\theta^{j+1} - 1).$$

Then by (7.20), it's trivial to establish the following iterative relation:

$$I(r_j, s_j) \leq \beta_7 \beta_8 \theta^{j m_1} \rho^{-n(\theta-1)} \left[I(r_{j-1}, s_{j-1})\right]^\theta, \text{ for every} j \in \mathbb{N}_0,$$

where the positive constant β_8 depends only on n, p_1, t_1 and $t_\star$.

Using this recurrent relation, we obtain that for every $j \in \mathbb{N}$

$$(7.21) \qquad I(r_j, s_j) \leq \left[\beta_9 \rho^{-n} I(r_0, s_0)\right]^{\theta^j},$$

where the positive constant β_9 depends only on $n, p_1, c_1, c_2, c_7, c, \hat{c_1}, \overline{c}, t_1, t_\star, \text{diam}\Omega$,

$M_1, M_2, \gamma(u_2, f_1), ||1/\nu_1||_{L^{t_1}(\Omega)}$ and χ_n .

We note that due to Hypothesis 2.1, $r_0 > 1$.

Now let us estimate $I(r_0, s_0)$. To this aim, we assume that

$$(7.22) \qquad meas\left\{x \in B\left(y, \frac{3}{2}\rho\right) : \ u_1(x) \geq \frac{\omega_1 + \omega_2}{2}\right\} \geq \frac{1}{2} \ measB\left(y, \frac{3}{2}\rho\right).$$

According to lemma 4 of [1], we deduce

$$\int_{B(y,\frac{3}{2}\rho)} (\lg G_1)^{r_0} dx \leq \beta_{10}\rho^n + \frac{\beta_{10}\rho}{2e\omega}\int_{B(y,\frac{3}{2}\rho)} \sum_{i=1}^n \left|\frac{\partial u_1}{\partial x_i}\right| (\lg G_1)^{r_0 - 1} G_1 dx,$$

where β_{10} depends only on n, p_1 and t_1.

By means of Young inequality and last inequality we get

$$(7.23) \quad \int_{B(y,\frac{3}{2}\rho)} (\lg G_1)^{r_0} dx \leq r_0 \beta_{10}\rho^n + \left(\frac{\beta_{10}\rho}{2e\omega}\right)^{r_0} \int_{B(y,\frac{3}{2}\rho)} \left\{\sum_{i=1}^n \left|\frac{\partial u_1}{\partial x_i}\right|\right\}^{r_0} G_1^{r_0} dx.$$

Using Hölder inequality and (7.13), we obtain

$$\int_{B(y,\frac{3}{2}\rho)} \left\{ \sum_{i=1}^{n} \left| \frac{\partial u_1}{\partial x_i} \right| \right\}^{r_0} G_1^{r_0} dx \leq$$

(7.24)

$$\leq \beta_{11}(2e\omega)^{r_0} \left\{ \int_{B(y,2\rho)} \left(\frac{1}{\nu_1} \right)^{t_1} dx \right\}^{1/(t_1+1)} \left\{ \int_{B(y,2\rho)} \Psi_1 dx \right\}^{t_1/(t_1+1)},$$

where the positive constant β_{11} depends only on $n, p_1, p_2, c_1, c_2, c_7, t_1, \operatorname{diam}\Omega$, and M_1 , M_2.

From (7.18) we get

$$\text{(7.25)} \quad \left\{ \int_{B(y,2\rho)} \left(\frac{1}{\nu_1} \right)^{t_1} dx \right\}^{1/(t_1+1)} \left\{ \int_{B(y,2\rho)} \Psi_1 dx \right\}^{t_1/(t_1+1)} \leq \beta_{12}\rho^{n-r_0},$$

where the positive constant β_{12} depends only on $n, p_1, t_1, t_\star, \gamma(u_2, f_1), ||1/\nu_1||_{L^{t_1}(\Omega)}$ and χ_n.

Finally, due to the definitions of the integrals $I(r,s)$ and properties of the function φ_1 we have

$$I(r_0, s_0) \leq \int_{B(y,\frac{3}{2}\rho)} (\lg G_1)^{r_0} dx.$$

From (7.23)-(7.25) and last inequality it follows that

$$I(r_0, s_0) \leq \beta_{13}\rho^n,$$

where the positive constant β_{13} depends only on $n, p_1, t_1, t_\star, M_1, M_2, c_1, c_2, c_7, c, \gamma(u_2, f_1)$,

$||1/\nu_1||_{L^{t_1}(\Omega)}$ and χ_n.

Last inequality and (7.21) imply

$$I(r_j, s_j) \leq [\beta_9\beta_{13}]^{\theta^j}, \text{ for every } j \in \mathbb{N}.$$

Then, we can conclude that

$$\operatorname*{ess\,sup}_{B(y,\rho)} G_1(x) \leq (1 + \beta_9\beta_{13})$$

and, so,

$$osc\{u_1, B(y,\rho)\} \leq (1 - e^{-\beta_9\beta_{13}})\omega + \rho^{a_1}.$$

Recall that we proved (7.1) under assumption (7.22). If (7.22) is not true, we take instead of G_1 the function $G_2 = \frac{2e\omega}{\omega_2 - u_1 + \rho^{a_1}}$ in $B(y, 2\rho)$ and, $G_2 = e$, in $\Omega \setminus B(y, 2\rho)$, and arguing as above, we establish (7.1) again.

Now from (7.1), taking into account Lemma 4.8 of [3], Chapter 2, we deduce that there exist positive constants C' and $\sigma'(< 1)$ depending on known values such that

$$osc\{u_1, B(y,\rho)\} \leq C'[d(y,\partial\Omega)]^{-\sigma'}\rho^{\sigma'}, \qquad \text{for every } \rho \in]0, d(y,\partial\Omega)].$$

Thus, for every open set Ω', $\overline{\Omega'} \subset \Omega$, and every $x', x'' \in \Omega'$,

$$|u(x') - u(x'')| \leq C'[d(\Omega', \Omega)]^{-\sigma'}|x' - x''|^{\sigma'}.$$

Analogous property we establish for the function u_2 using considerations which are similar to the above given.

In this way we achieved the Hölder continuity of generalized solution (u_1, u_2) for the Dirichlet problem of system (1.1) in the interior of Ω.

8 An Example

First of the all, we consider an example of the functions $a_i^{(j)}$, $g^{(j)}$, $i = 1, 2, ..., n$, $j = 1, 2$, satisfying conditions of Section 3. Let $p_j \in \mathbb{R}$ $(j = 1, 2)$, $n \in \mathbb{N}$, $n \geq 3$, such that $2 < p_j < n$.

Let $k > 0$, and let for every $i = 1, 2, ..., n$, $A_i^{(1)}$, $A_i^{(2)}$, $A_i^{(3)}$, $A_i^{(4)}$ be functions on Ω such that

(8.1) $\quad 0 \leq A_i^{(1)} \leq k\nu_1^{2/p_1}$,

(8.2) $\quad |A_i^{(2)}| \leq k\nu_1^{1/p_1}\nu_2^{1/p_2}$,

(8.3) $\quad |A_i^{(3)}| \leq k\nu_1^{1/p_1}\nu_2^{1/p_2}$,

(8.4) $\quad 0 \leq A_i^{(4))} \leq k\nu_2^{2/p_2}$,

(8.5) $\quad (A_i^{(2)} + A_i^{(3)})^2 \leq 4A_i^{(1)}A_i^{(4)}$

in Ω.

Let for every $i = 1, 2, ..., n$, and every $x \in \Omega$, $\eta^{(1)}$, $\eta^{(2)} \in \mathbb{R}^n$,

$$a_i^{(1)}(x, \eta^{(1)}, \eta^{(2)}) = \nu_1(x)|\eta_i^{(1)}|^{p_1-2}\eta_i^{(1)} + A_i^{(1)}(x)\eta_i^{(1)} + A_i^{(2)}(x)\eta_i^{(2)},$$

$$a_i^{(2)}(x, \eta^{(1)}, \eta^{(2)}) = \nu_2(x)|\eta_i^{(2)}|^{p_2-2}\eta_i^{(2)} + A_i^{(3)}(x)\eta_i^{(1)} + A_i^{(4)}(x)\eta_i^{(2)}, \qquad .$$

Define

(8.6) $\quad \overline{p}_1 = \dfrac{p_2}{p_2 - 1}$, $\overline{p}_2 = \dfrac{p_1}{p_1 - 1}$.

Since $p_j > 2$, $j = 1, 2$, we have $\overline{p}_j \in (0, p_j)$, $j = 1, 2$.

From (8.1)-(8.4) it follows that the functions $a_i^{(j)}$ satisfy inequalities (3.1)-(3.4) with some positive constants c_α, $\alpha = 1, 2, 4, 5, 7, 8$, depending only on n, p_1, p_2, k.

Moreover, from (8.5) it follows that the functions $a_i^{(j)}$ satisfy inequality (3.5).

Now let b_1, b_2, b_3, b_4 be numbers such that $b_1, b_4 > 0$ and $(b_2 + b_3)^2 \leq 4b_1b_4$, and let for every $x \in \Omega$ and every $u_1, u_2 \in \mathbb{R}$,

$$g^{(1)}(x, u_1, u_2) = |u_1|^{p_1-2}u_1 + b_1u_1 + b_2u_2,$$

$$g^{(2)}(x, u_1, u_2) = |u_2|^{p_2-2}u_2 + b_3u_1 + b_4u_2.$$

It easy to check that the functions $g^{(j)}$ satisfy equality (3.6) and inequality (3.7) with a positive constant c_3 depending only on p_1, p_2 and b_1, b_2, b_3, b_4. Moreover, the function $g^{(1)}$ satisfies inequality (3.8) with $\sigma_2 = 2$ and $c_9 = b_2^2/b_1$, and the function $g^{(2)}$ satisfies inequality (3.9) with $\sigma_1 = 2$ and $c_6 = b_3^2/b_4$. These functions satisfy inequality (3.10) as well.

Now, we give an example where Hypotheses 2.1, 6.1 and 7.1 are satisfied.

Suppose for simplicity that $0 \in \partial\Omega$ and, additionally, we assume that

$$p_j > \frac{n}{2}, \ (j = 1, 2).$$

Let $0 < \gamma_j < \dfrac{n}{2}\dfrac{p_j - n/2}{3n/2 - p_j}$, $(j = 1, 2)$, and let for every $j = 1, 2$ the function $\nu_j : \Omega \to \mathbb{R}$ be defined by

$$\nu_j(x) = |x|^{\gamma_j}.$$

For every $j = 1, 2$, let t_j be such that

$$\frac{n}{p_j - n/2} < t_j < 1 + \frac{n}{2\gamma_j}.$$

It results $\frac{n}{p_j} < t_j < \frac{n}{\gamma_j}$, then the function $\nu_j(x)$ $(j = 1, 2)$, satisfies the Hypothesis 2.1. Moreover, it easy to verify that

$$|x|^{2\gamma_j} \in A_{1+\frac{1}{t_j-1}}$$

then, Hypothesis 7.1 holds with $\bar{t}_j = 2$, $(j = 1, 2)$.

Finally, if the number p_j $(j = 1, 2)$ is such that $p_j > 6$ then conditions of Hypothesis 6.1 are satisfied with $\sigma_2 = 2$ and $\overline{p}_j$, $j = 1, 2$, defined by (8.6).

9 Conclusion

In this paper, we have studied solvability and regularity properties of solutions to the system of equations:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^{(j)}(x, \nabla u_1, \nabla u_2) + g^{(j)}(x, u_1, u_2) = f^{(j)}(x) \quad \text{in } \Omega, \; j = 1, 2,$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n > 2$. In particular, we have obtained that any solutions is locally Hölder continuous in Ω. We observe that this could be the first step to getting Hölder regularity up to the boundary of Ω.

Competing Interests

Author has declared that no competing interests exist.

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