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Dynamics of Capital-labour Model with Hattaf-Yousfi Functional Response

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

The labour force is a fundamental component of every modern economy. It is also called the workforce that is the total number of the people who are eligible to work, including the employed and unemployed people. The company supplies free jobs which number is proportional to the invested capital. In this work, we propose a mathematical model that describes the dynamics of free jobs and labour force. In the model, the rate by which the labour force is filling in free jobs is modeled by Hattaf-Yousfi functional response. Furthermore, we first show that the proposed model is mathematically and economically well-posed. Moreover, the dynamical behavior of the model is studied by determining the existence and stability of equilibria.

Keywords: Capital-labour model; free jobs; labour force; Hattaf-Yousfi functional response.

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1 Introduction and Presentation of the Model

The labour supply and labour demand are the essential components governing in the labour market, which are influenced by the gross domestic product and demographic factors that vary across households. In our context, the labour supply is represented by the number of free jobs and demand by the labour force. Note that the labour force or workforce is the total number of the people who are eligible to work, including the employed and unemployed people.

To study the dynamics of labour market, we propose the following capital-labour model

$$\begin{cases} \dfrac{du(t)}{dt} = ru(t)\left(1 - \frac{u(t)}{K}\right) - \dfrac{mu(t)v(t)}{\alpha_0 + \alpha_1 u(t) + \alpha_2 v(t) + \alpha_3 u(t)v(t)}, \\ \dfrac{dv(t)}{dt} = \dfrac{mu(t)v(t)}{\alpha_0 + \alpha_1 u(t) + \alpha_2 v(t) + \alpha_3 u(t)v(t)} - dv(t), \end{cases} \tag{1.1}$$

where $u(t)$ denotes the number of free jobs at time t and $v(t)$ represents the total labour force, i.e., the number of those employed and unemployed at time t. The positive constant r is the natural per of capita growth of free jobs and $K > 0$ is its theoretical eventual maximum of the number of free jobs (related to the theoretical maximum of investment capital). The positive parameter d is the death rate of labour force. In the model, the rate by which the labour force filling in the free jobs is modeled by Hattaf-Yousfi functional response [1] of the form $\dfrac{mu}{\alpha_0 + \alpha_1 u + \alpha_2 v + \alpha_3 uv}$, where the positive coefficient m is the maximum growth of labour force and α_0, α_1, α_2, α_3 are nonnegative constants. It is important to note that this functional response generalizes many functional responses existing in the literature such as the Beddington-DeAnglis functional response [2, 3] when $\alpha_0 = 1$ and $\alpha_3 = 0$, the Crowley-Martin functional response [4] when $\alpha_0 = 1$ and $\alpha_3 = \alpha_1\alpha_2$, and the specific functional response introduced by Hattaf et al. (see Section 5, [5]) when $\alpha_0 = 1$.

On the other hand, the models without cross-diffusion presented by Balázsi and Kiss [6] are special cases of system (1.1). In the fact, when $\alpha_0 = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = 0$, we get the first simple market model of [6]. When $\alpha_0 = \alpha_3 = 0$ and $\alpha_1 = 1$, we obtain the second model of [6] with Holling-type ratio-dependent response.

The organization of this paper is as follows. The next section deals with well posedness and equilibria of system (1.1). In Section 3, we investigate the stability of equilibria. Finally, an application of our results is given in Section 4.

2 Well Posedness and Equilibria

In this section, we first show that our model (1.1) is mathematically and economically well posed. After, we determine the equilibria of (1.1).

Proposition 2.1. *All solutions of system (1.1) starting from nonnegative initial conditions, remain positive and bounded for all $t > 0$. Moreover, we have*

$$\limsup_{t\to+\infty} N(t) \leq \frac{rK}{\mu},$$

where $N(t) = u(t) + v(t)$ and $\mu = \min(r, d)$.

Proof. For the nonnegativity, we show that any solution starting in the first quadrant $\mathbf{R}^2_+ = \{(x, y) \in \mathbf{R}^2 : x \geq 0,\ y \geq 0\}$ remains there forever.

From (1.1), we have

$$\begin{aligned} u(t) &= u(0)\exp\left(\int_0^t \big(r(1-\frac{u(t)}{K}) - \frac{mv(t)}{\alpha_0+\alpha_1 u(t)+\alpha_2 v(t)+\alpha_3 u(t)v(t)}\big)dt\right) \geq 0, \\ v(t) &= v(0)\exp\left(\int_0^t \big(\frac{mu(t)}{\alpha_0+\alpha_1 u(t)+\alpha_2 v(t)+\alpha_3 u(t)v(t)} - d\big)dt\right) \geq 0, \end{aligned}$$

Hence, the nonnegativity of all solutions initiating in $\mathbf{R}^2_+$ is guaranteed.

Now, we prove the boundedness of solutions. From system (1.1), we have

$$\begin{aligned} \frac{dN}{dt} &= ru(t)\left(1-\frac{u(t)}{K}\right) - dv \\ &= -\frac{r}{K}\left[(u-K)^2 + Ku - K^2\right] - dv \\ &\leq rK - (ru + dv) \\ &\leq rK - \mu N. \end{aligned}$$

Then $\limsup_{t\to+\infty} N(t) \leq \frac{rK}{\mu}$, which implies that $u(t)$ and $v(t)$ are bounded. ■

Next, we study the existence of equilibria of system (1.1). It is easy to see that $E_0(0,0)$ and $E_1(K,0)$ are two trivial equilibria of (1.1). To find the other equilibrium of (1.1), we solve the following system

$$ru\left(1-\frac{u}{K}\right) - \frac{muv}{\alpha_0+\alpha_1 u+\alpha_2 v+\alpha_3 uv} = 0, \quad (2.1)$$

$$\frac{muv}{\alpha_0+\alpha_1 u+\alpha_2 v+\alpha_3 uv} - dv = 0. \quad (2.2)$$

From (2.1) and (2.2), we get $v = \dfrac{ur(K-u)}{dK}$ and

$$\alpha_3 ru^3 + (\alpha_2 - \alpha_3 K)ru^2 + (m - d\alpha_1 - \alpha_2 r)Ku - Kd\alpha_0 = 0. \quad (2.3)$$

Since $v = \frac{ur(K-u)}{dK} \geq 0$, we have $u \leq K$. Hence, there is no economic equilibrium when $u > K$. Consider the function f defined on interval $[0, K]$ by

$$f(u) = \alpha_3 ru^3 + (\alpha_2 - \alpha_3 K)ru^2 + (m - d\alpha_1 - \alpha_2 r)Ku - Kd\alpha_0. \quad (2.4)$$

Obviously, if $\alpha_0 = 0$, Eq. (2.3) admits two solutions one is trivial and the other exists if $m - d\alpha_1 - r\alpha_2 < 0$. When $\alpha_0 > 0$, we have $f(0) = -Kd\alpha_0 < 0$ and $f(K) = dK(\alpha_1 K + \alpha_0)(T_0 - 1)$, where T_0 is defined by

$$T_0 = \frac{mK}{d(\alpha_0 + \alpha_1 K)}.$$

If $T_0 > 1$, Eq. (2.3) admits at least one solution $u^* \in (0, K)$ which implies that system (1.1) has at least one capital-labour equilibrium $E^*(u^*, v^*)$ with $v^* = \frac{u^* r(K-u^*)}{dK}$.

To determine the uniqueness of the solution u^*, we discuss the following cases:

- If $\alpha_3 = \alpha_2 = 0$, then Eq. (2.3) has a unique positive solution given by

$$u^* = \frac{d\alpha_0}{m - d\alpha_1}.$$

- If $\alpha_3 = 0$ and $\alpha_2 \neq 0$, then Eq. (2.3) has a unique positive solution given by

$$u^* = \frac{-(m - d\alpha_1 - \alpha_2 r)K + \sqrt{(m - d\alpha_1 - \alpha_2 r)^2 K^2 + 4Kd\alpha_0\alpha_2 r}}{2\alpha_2 r}.$$

- If $\alpha_3 \neq 0$, by applying the rule signs of Descartes, Eq. (2.3) has a unique positive root u^*, if any of the following three conditions is satisfied

$$\alpha_2 - \alpha_3 K > 0 \textit{ and } m - d\alpha_1 - \alpha_2 r > 0, \tag{2.5}$$

$$\alpha_2 - \alpha_3 K > 0 \textit{ and } m - d\alpha_1 - \alpha_2 r < 0, \tag{2.6}$$

$$\alpha_2 - \alpha_3 K < 0 \textit{ and } m - d\alpha_1 - \alpha_2 r < 0. \tag{2.7}$$

By using Cardan's formula, u^* is given by

$$u^* = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \frac{c_1}{3}, \tag{2.8}$$

where $c_1 = \frac{\alpha_2}{\alpha_3} - K$, $c_2 = \frac{(m-d\alpha_1-\alpha_2 r)K}{\alpha_3 r}$, $c_3 = \frac{-Kd\alpha_0}{\alpha_3 r}$, $p = c_2 - \frac{c_1^2}{3}$ and $q = \frac{c_1}{27}(2c_1^2 - 9c_2) + c_3$.

3 Stability of Economic Equilibria

The trivial equilibrium $E_0(0,0)$ represents the absence of both free jobs and labour force. Therefore, this equilibrium is not important in economy. For an arbitrary equilibrium $E(u,v)$, the characteristic equation is given by

$$\begin{vmatrix} r(1-\frac{2u}{k}) - \frac{mv(\alpha_0+\alpha_2 v)}{(\alpha_0+\alpha_1 u+\alpha_2 v+\alpha_3 uv)^2} - \lambda & -\frac{mu(\alpha_0+\alpha_1 u)}{(\alpha_0+\alpha_1 u+\alpha_2 v+\alpha_3 uv)^2} \\ \frac{mv(\alpha_0+\alpha_2 v)}{(\alpha_0+\alpha_1 u+\alpha_2 v+\alpha_3 uv)^2} & \frac{mu(\alpha_0+\alpha_1 u)}{(\alpha_0+\alpha_1 u+\alpha_2 v+\alpha_3 uv)^2} - d - \lambda \end{vmatrix} = 0. \tag{3.1}$$

Theorem 3.1. *Let us define* $T_0 = \frac{mK}{d(\alpha_0+\alpha_1 K)}$.

The trivial equilibrium $E_1(K,0)$ *is locally asymptotically stable if* $T_0 < 1$ *and it is unstable if* $T_0 > 1$.

Proof. At E_1, Eq.(3.1) becomes

$$(r + \lambda)\left(\frac{mK}{\alpha_0 + \alpha_1 K} - d - \lambda\right) = 0, \tag{3.2}$$

where the roots are: $\lambda_1 = -r$, $\lambda_2 = d(T_0 - 1)$. It is clear that λ_1 is negative. Moreover, λ_2 is negative when $T_0 < 1$ and it is positive if $T_0 > 1$. This completes the proof. ■

Theorem 3.1 only establishes the local stability of E_1. However, the following theorem establishes its global stability.

Theorem 3.2. *If* $T_0 \leq 1$, *then the trivial equilibrium* E_1 *is globally asymptotically stable.*

Proof. Consider the following Lyapunov functional

$$V(t) = u(t) - K - \int_K^{u(t)} \frac{g(K,0)}{g(x,0)} dx + v(t), \tag{3.3}$$

where $g(u,v) = \dfrac{mu}{\alpha_0 + \alpha_1 u + \alpha_2 v + \alpha_3 uv}$.

Calculating the time derivative of V along the positive solution of system (1.1), we get

$$\begin{aligned}\dot{V}(t)|_{(1.1)} &= ru\left(1-\frac{g(K,0)}{g(u,0)}\right)\left(1-\frac{u}{K}\right)+\frac{g(K,0)}{g(u,0)}g(u,v)v-dv,\\ &= ru\left(1-\frac{g(K,0)}{g(u,0)}\right)\left(1-\frac{u}{K}\right)+dv\left(\frac{g(u,v)}{g(u,0)}T_0-1\right),\\ &\le ru\left(1-\frac{g(K,0)}{g(u,0)}\right)\left(1-\frac{u}{K}\right)+(T_0-1)dv.\end{aligned}$$

Since g is increasing function with respect u, we have

$$\left(1-\frac{g(K,0)}{g(u,0)}\right)\left(1-\frac{u}{K}\right)\le 0.$$

Since $T_0 \le 1$, we have $\dot{V}(t)|_{(1.1)} \le 0$. Further, $\dot{V}(t)|_{(1.1)} = 0$ if and only if $u = K$ and $v = 0$. Then the largest compact invariant set in $\Gamma = \{(u,v)|\dot{V} = 0\}$ is just the singleton $\{E_1\}$. From LaSalle invariance principle [7], we deduce that E_1 is globally asymptotically stable. ■

Finally, we focus on the local stability of the capital-labour equilibrium E^*. Evaluating (3.1) at E^* and computing the characteristic equation about this point, we have

$$\lambda^2 + a_1\lambda + a_2 = 0, \tag{3.4}$$

where

$$\begin{aligned}a_1 &= d-r\big(1-\frac{2u^*}{K}\big)+\frac{d}{mu^*}\big[r\big(1-\frac{u^*}{K}\big)\big(\alpha_0+\alpha_2v^*\big)-d(\alpha_0+\alpha_1u^*)\big]\\ a_2 &= \frac{d}{mu}\big[r\big(1-\frac{2u^*}{K}\big)\big(d(\alpha_0+\alpha_1u^*)-mu^*\big)+dr(1-\frac{u^*}{K})(\alpha_0+\alpha_2v^*)\big].\end{aligned}$$

Therefore, we have the following result.

Theorem 3.3. *Assume that $T_0 > 1$. If $a_1 > 0$ and $a_2 > 0$, then the capital-labour equilibrium $E^*(u^*, v^*)$ is locally asymptotically stable.*

4 Application

The aim of this section is to apply our main results to the following capital labour model

$$\begin{cases}\dfrac{du(t)}{dt} = ru(t)\left(1-\frac{u(t)}{K}\right)-mu(t)v(t),\\ \dfrac{dv(t)}{dt} = mu(t)v(t)-dv(t),\end{cases} \tag{4.1}$$

which is a special case of system (1.1) by letting $\alpha_0 = 1$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$. This model was proposed by Balázsi and Kiss [6]. In this case, $T_0 = \frac{mK}{d}$, $E^*\big(\frac{d}{m}, \frac{r}{mT_0}(T_0-1)\big)$, $a_1 = \frac{dr}{Km}$ and $a_2 = \frac{dr}{T_0}(T_0-1)$. In [6], Balázsi and Kiss only determined the local stability of E^*. By applying Theorems 3.1, 3.2 and 3.3, we get the following corollary.

Corollary 4.1.

(i) *If $T_0 \le 1$, then the trivial equilibrium E_1 of system (4.1) is globally asymptotically stable.*

(ii) *If $T_0 > 1$, then the trivial equilibrium E_1 becomes unstable and the capital-labour equilibrium E^* of system (4.1) is locally asymptotically stable.*

Now, we simulate system (4.1) with the following parameter values: $r = 1$, $K = 100$, $m = 0.01$ and $d = 0.2$. By a simple calculation, we get $E^*(20, 80)$ and $T_0 = 5 > 1$. By Corollary 4.1 (ii), we deduce that E^* is locally asymptotically stable. Fig. 1 illustrates this observation.

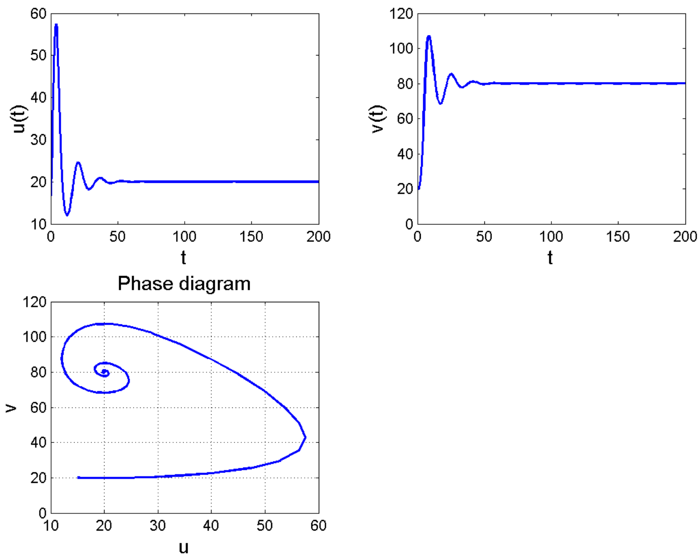

Fig. 1. Stability of the capital-labour equilibrium E^*

5 Conclusion

In this work, we have proposed and analyzed a capital-labour model with Hattaf-Yousfi functional response. This functional response modeled the rate by which the labour force is filling in free jobs and it covers various types existing in the literature such as Beddington-DeAngelis response and Crowley-Martin response. Firstly, we have proved that proposed model is mathematically and economically well-posed. In addition, the stability of equilibria are established in terms of a threshold value T_0. More precisely, the trivial equilibrium $E_1(K, 0)$ is globally asymptotically stable if $T_0 \leq 1$. When $T_0 > 1$, E_1 becomes unstable and the capital-labour equilibrium E^* is locally asymptotically stable under some hypotheses ($a_1 > 0$ and $a_2 > 0$).

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Competing Interests

Authors have declared that no competing interests exist.

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