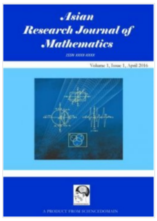

Asian Research Journal of Mathematics

9(4): 1-9, 2018; Article no.ARJOM.41715

ISSN: 2456-477X

S-Strongly Finite Type Rings

Mosbah Eljeri[1,2*]

[1] *Monastir University, Tunisia.*

[2] *Majmaah University, Kingdom of Saudi Arabia.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2018/41715

Editor(s):

(1) Junjie Chen, Department of Electrical Engineering, University of Texas at Arlington, USA.

Reviewers:

(1) Areej M. Abduldaim, University of Technology, Iraq.

(2) Celil Nebiyev, 19 Mayis University, Turkey.

Complete Peer review History: http://www.sciencedomain.org/review-history/25090

Received: 27th March 2018

Accepted: 3rd June 2018

Published: 11th June 2018

Original Research Article

Abstract

In this paper a ring means a commutative ring with identity element and S is a multiplicatively closed subset of the studied ring whose elements are regular. Inspiring from the work of J. Arnold about strongly finite type rings and D. E. Rush's about noetherian spectrum rings, we introduce two types of rings that are S-strongly finite type rings and S-noetherian spectrum rings. We give some characterizations of these rings and we illustrate them by many examples.

Keywords: Commutative rings; S-finite ideal; S-SFT ring.

MSC: 13A15, 13F25, 13EXX.

1 Introduction

In this work R is a commutative ring with identity, I is an ideal and S is a multiplicative (closed under multiplication) subset of R whose elements are regular. Many classic concepts from ideal theory are generalized to S-concepts for instance see [1], [2], [3], [4], [5]. In [6], Anderson et al. defined the ideal I to be an S-finite ideal, if there exists an element s of S and a finitely generated ideal J satisfying: $sI \subseteq J \subseteq I$. When J is principal, I is said to be S-principal. Then they defined the S-noetherian (respectively, the S-principal) ring to be the ring whose all ideals are S-finite (

Corresponding author: E-mail: elljeri@yahoo.fr, m.eljeri@mu.edu.sa

respectively, S-principal). In [7], Arnold introduced the concept of strongly finite type ideal of R. The ideal I is said to be a *strongly finite type* ideal (shortly, SFT ideal) if there exists a finitely generated ideal $J \subseteq I$ and $n \in \mathbb{N}^*$ such that $x^n \in J, \forall x \in I$. A ring whose all ideals are SFT ideals is called a strongly finite type ring (shortly, SFT ring). In the same mode, We define the ideal I to be an S-Strongly finite type ideal (in short S-SFT ideal) if there exists an S-finite ideal $J \subseteq I$ and $n \in \mathbb{N}^*$ such that $x^n \in J, \forall x \in I$. A ring whose all ideals are S-SFT ideals is called an S-strongly finite type ring (in short, S-SFT ring). Many characterizations and examples for this type of rings are given. We show that a homomorphic image of an S-SFT ring is an $f(S)$-SFT ring, f is the ring homomorphism. We generalize the results of Arnold of [7] and for SFT ideals and SFT-rings to S-SFT cases. We introduce the concept of S-radical ideal and we define the S-radical of an ideal. We introduce and study the class of S-noetherian spectrum rings.

2 S-SFT rings

Definition 2.1. *Let R be a commutative ring and S be a multiplicative subset of it. Let I be an ideal of R.*

1. *[According to [6]] We say that I is an S-finite ideal if there exist a finitely generated ideal J of R and an $s \in S$ satisfying $sI \subseteq J \subseteq I$.*
2. *We say that I is an S-strongly finite type ideal , in short S-SFT ideal, if it contains an S-finite ideal F of R and there exists $n \in \mathbb{N}^*$ such that $\forall x \in I, x^n \in F$.*

Remark 2.2. *Let R, I and S be as in the Definition 2.1.*

1. *If I is S-finite or SFT ideal then it is an S-SFT ideal.*
2. *If T is a multiplicative set containing S then I is S-SFT ideal implies that it is a T-SFT ideal.*
3. *I is an SFT ideal if and only if it is an $U(R)$-SFT ideal if and only if it is an E-SFT ideal for some multiplicative E set of $U(R)$. $U(R)$ denotes the units set of R.*
4. *If $I \cap S \neq \emptyset$ then I is a S-SFT ideal.*

Proof:

1. Immediately, from Definition 2.1.
2. if I is an SFT ideal then there exists $n \in \mathbb{N}^*$ and $J \subseteq I$ finitely generated such that $\forall x \in I,\ x^n \in J$. Since J is finitely generated then it is S-finite and satisfies 2.1.2. Then I is an S-SFT ideal.
3. Since each S-finite ideal is a T-finite ideal, we get immediately this result.
4. It is obvious that an ideal J is finitely generated iff it is $U(R)$-finite iff it is E-finite because $sJ = J, \forall s \in U(R)$.

Definition 2.3. *Let R be a commutative ring and S a multiplicative set of it.*

1. *[According to [6]] R is said to be S-noetherian if all its ideals are S-finite.*
2. *We define R to be S-strongly finite type ring (in short S-SFT ring) if all its ideals are S-SFT ideals.*

Immediately from Remark 2.2, we have.

Remark 2.4. *Let R, I and S be as in Definition 2.1.*

1. *If R is an S-noetherian or an SFT ring then it is an S-SFT ring.*
2. *If T is a multiplicative set containing S then all S-SFT rings are T-SFT rings.*
3. *R is an SFT ring if and only if it is an $U(R)$-SFT ring if and only if it is an E-SFT ring for some multiplicative set E of $U(R)$.*

Lemma 2.5. *Let R, I and S be as in Definition 2.1. For $n \in \mathbb{N}^*$ we denote by (I, n) the ideal of R that is generated by $\{x^n, x \in I\}$. The following are equivalent:*

1. *I is an S-SFT ideal.*
2. *There exists $n \in \mathbb{N}^*$ such that (I, n) is S-finite.*
3. *There exists $n \in \mathbb{N}^*, s \in S$ and $J \subseteq I$ finitely generated such that $\forall x \in I, sx^n \in J$.*

Proof: 1.$\Rightarrow$ 2. If I is an S-SFT ideal, then $\exists s \in S, n \in \mathbb{N}^*, F$ an ideal and J finitely generated such that $sF \subseteq J \subseteq F \subseteq I$ and $x^n \in F, \forall x \in I$. Then $(I, n) \subseteq F$ and we get $s(I, n) \subseteq J \subseteq (I, n)$, hence (I, n) is S-finite.
2.$\Rightarrow$ 3. , 3.$\Rightarrow$ 2 . and 2.$\Rightarrow$ 1. are immediate.

Example 2.6. Let K be a field , $\{X_i, i \in \mathbb{N}\}$ be a set of indeterminates over K. Let $R := K[X_i, i \in \mathbb{N}]$, S be a multiplicative set of it and $M = (X_i, i \in \mathbb{N})$ the ideal of polynomials with zero constant term. Then M is an S-SFT ideal iff $M \cap S \neq \emptyset$. The if part is obvious. For the only if part, take S such that $M \cap S = \emptyset$ and suppose that M is an S-SFT ideal. Then there exists $f_1, ..., f_r \in M, n \in \mathbb{N}^*$ *and* $s \in S$ $/\forall f \in M, sf^n \in (f_1, ..., f_r)$. Since $s, f_1, ..., f_r$ are polynomials then they use only finite number of variables, let them be $X_1, ..., X_t$. $X_{t+1} \in M$ then $sX^n_{t+1} = P(X_1, ..., X_t) \in (f_1, ..., f_r)$. Since $P(X_1, ..., X_t) \in M$, then its constant term is zero then , by taking $X_1 = ... = X_t = 0$ we get $s(0, ..., 0)X^n_{t+1} = P(0, ..., 0) = 0$.That is $s(0, ..., 0) = 0$, then s has zero constant term. Consequently, $s \in M$ then we get a contradiction, since $M \cap S = \emptyset$.

Definition 2.7. *If S is a multiplicative set of R, the set $\overline{S} := \{x \in R/\exists y \in R, xy \in S\}$ is called the saturation of S.*

Proposition 2.8.

1. *An ideal I is an S-SFT ideal if and only if it is an $\bar{S}$-SFT ideal.*
2. *A ring R is an S-SFT ring if and only if it is $\bar{S}$-SFT ring.*

Proof:

1. Assume that I is an S-SFT ideal. Since $S \subseteq \bar{S}$, by Remark 2.2. 2, I is an $\bar{S}$-SFT ideal. Conversely, if I is an $\bar{S}$-SFT ideal then, Lemma 2.5, there exist $s \in \bar{S}, n \in \mathbb{N}^*$ and a finitely generated ideal $J \subseteq I$ such that $\forall x \in I, sx^n \in J$. $s \in \bar{S}$ then $\exists y \in R/\ ys \in S$ and then $\forall x \in I, (ys)x^n \in J$. Consequently, I is S-SFT ideal.
2. immediatly from 1.

Theorem 2.9. *Let A , B be two rings, S be a multiplicative set of A and $f : A \longrightarrow B$ be a ring homomorphism such that for all $x \in S$, $f(x)$ is regular in the ring $f(A)$. If A is an S-SFT ring then $f(A)$ is an $f(S)$-SFT ring.*

Proof: It is clear that $f(S)$ is multiplicative set of the ring $f(A)$. If H is an ideal of $f(A)$ then $f^{-1}(H) := \{x \in A/\ f(x) \in H\}$ is an ideal of A. Then it is an S-SFT ideal. By [Lemma 2.5], there exists $s \in S, n \in \mathbb{N}^*, J \subseteq f^{-1}(H)$ finitely generated ideal such that $sx^n \in J, \forall x \in f^{-1}(H)$. Then $\forall y \in H, \exists x \in f^{-1}(H)/\ y = f(x)$. We have $f(s)f(x)^n \in f(J) \subseteq H$ since $f(J)$ is finitely generated then H is an $f(S)$-SFT ideal.

Corollary 2.10. *Let R be a ring and S be a multiplicative set of it.*

Let I be an ideal of R disjoint from S . If R is an S-SFT ring then R/I is an S/I-SFT ring. Here $S/I = \{s + I, s \in S\}$.

Proof: 1. It is clear that R/I is a homomorphic image of R and the same homomorphism send S to S/I. According to Theorem 2.9, we get the result.

A valuation ring is integral domain whose ideals are totally ordered by inclusion.

Proposition 2.11. *Let D be a valuation ring and S be a multiplicative set of it . Let I be an ideal disjoint from S. The following are equivalent:*

1. *I is an S-SFT ideal.*
2. *$I^2 \neq I$.*
3. *I is an SFT ideal*

Proof: 1.$\Rightarrow$ 2. Assume that I is an S-SFT ideal of D disjoint from S. Suppose $I^2 = I$. Since I is an S-SFT ideal, then there exist $s \in S$, $n \in \mathbb{N}^*$ and $J \subseteq I$ finitely generated such that $sx^n \in J, \forall x \in I$. D is a valuation ring then J is principal. Then there exists $s \in S$, $n \in \mathbb{N}^*$ and $a \in I$ such that $sx^n \in aD, \forall x \in I$ then $sI^n \subseteq aD$. Since $s \notin I$, then $sI \subseteq aD \subseteq I \subseteq sD(*)$ then $sI^2 \subseteq aI \subseteq I^2 \subseteq sI$ then $sI = aI = I$. By (*), $I = aD$ then $aD = a^2D$. Since D is a domain then we get a invertible and then $I = D$, contradiction. Then $I^2 \neq I$.

2.$\Rightarrow$ 3. if $I^2 \neq I$ let $a \in I \setminus I^2$ then $I^2 \subseteq aD$ then I is an SFT ideal.
3.$\Rightarrow$ 1. is immediate.
The following is a generalization of Lemma 2.1 of [8].

Lemma 2.12. *Let I_1 and I_2 be two S-SFT ideals of a ring R. Let J be an ideal of R such that $I_1I_2 \subseteq J \subseteq I_1 \cap I_2$ then J is also an S-SFT ideal of R.*

Proof: Since I_i is an S-SFT ideal then there exist $s_i \in S$, $n_i \in \mathbb{N}^*$ and J_i a finitely generated ideal such that $\forall x \in I_i, s_ix^{n_i} \in J_i \subseteq I_i$. Consequently, $\forall x \in J, s_1s_2x^{n_1+n_2} \in J_1J_2 \subseteq I_1I_2 \subseteq J$. Since J_1J_2 is finitely generated we conclude that J is an S-SFT ideal.

Proposition 2.13. *R is an S-SFT ring if and only if all its prime ideals(disjoint from S) are S-SFT ideals*

Proof: Suppose that R is not an S-SFT ring. Then the set of non S-SFT ideals is nonempty. By Zorn Lemma there exists a maximal element P in the set of non S-SFT ideals. If P is not prime then there exists $x, y \in R \setminus P$ such that $xy \in P$ then $P + xR$ and $P + yR$ are S-SFT ideals. Since $(P + xR)(P + yR) \subseteq P \subseteq (P + xR) \cap (P + yR)$, then P is S-SFT, by Lemma 2.12. This means that P is prime. And this contradicts the fact all primes are S-SFT 's. We conclude that if all primes are S-SFT 's then so is R. The converse, is immediate.

Let R be a ring and K its total quotient ring. An overring $R_1$of R is a ring satisfying $R \subseteq R_1 \subseteq K$. R_1 is said to be a flat overring of R if it is flat as an R-module. The following definition is given in [9]. Let $\mathcal{S}$ be a multiplicative (i. e. multiplicatively closed) set of ideals of R and set $R_{\mathcal{S}} := \{x \in K/\ xA \subseteq R$ for some $A \in \mathcal{S}\}$. Then $R_{\mathcal{S}}$ is an overring of R called the $\mathcal{S}$-transform of R.

Theorem 2.14. *[Theorem 1.3 [9]] R_1 is flat overring of R iff $R_1 = R_T$, T is multiplicative set of ideals of R.*

Theorem 2.15. *Let R_1 be a flat overring of R. If R is an S-SFT ring then so is R_1. In particular, R_S is an S-SFT.*

Proof: Let Q be a prime ideal of R_1. then $P := Q \cap R$ is a prime ideal of R. Since R is S-SFT then so is P and then there exist an S-finite ideal $F \subseteq P$ of R and $n \in \mathbb{N}^*$ such that: $\forall p \in P, p^n \in F$.

Let $q \in Q \subseteq R_1 = R_T$, there exists $A \in T$ such $qA \subseteq R$ and then $qA \subseteq R \cap Q = P$. Then $\forall a \in A, q^n a^n \in F$. Let $J = \{x \in R_1 / q^n x \in FR_1\}$. It is easy to check that J is an ideal of R_1. Remark that $AR_1 \subseteq \sqrt{J}$. According to Theorem 2.14 $AR_1 = R_1$ and then $R_1 = \sqrt{J}$ so $R_1 = J$. Consequently, $q^n \in FR_1 \subseteq Q$. Since FR_1 is S-finite then Q is an S-SFT ideal of R_1.

Example 2.16. *If D is a non SFT domain and $S = D \setminus \{0\}$ then $D_S = q.f\{D\}$ is an SFT domain (noetherian) but D is not S-SFT (not SFT).* This example shows that the converse of Theorem 2.15 is false.

3 Ring with S-noetherian spectrum

As it is well known a Noetherian spectrum ring is a ring satisfying the ascending chain condition on radical ideals equivalently it is a ring whose primes are radicals of finite type ideals [See [10]]. In this paragraph, we generalize this concept to a S-concept and we give some results about it.

Notation: We denote by $Spec(R, S)$ the set of all prime ideals of R disjoint from S.

Definition 3.1. *Let S be a multiplicative set of the ring R and I be one of its ideals.*

1. *We define the S-radical of I and we denote it by $\sqrt[S]{I}$ as the following:*
   - $\sqrt[S]{I} = \bigcap_{I \subseteq P \in Spec(R,S)} P$, *if $S \cap I = \emptyset$.*
   - $\sqrt[S]{I} = R$ *otherwise.*
2. *I is said to be S-radical if $\sqrt[S]{I} = I$.*
3. *R is called S-noetherian spectrum ring if it satisfies the ascending chain condition (acc) on S-radicals.*
4. *I is said to be S-radically finite if there exists a finitely generated ideal $J \subseteq I$ of R such that $\sqrt[S]{I} = \sqrt[S]{J}$.*

$R[[X]]$ denotes the formal power series ring in one indetermite X with coefficients in R. If I is an ideal of R then $I[[X]]$ is the ideal whose elements have coefficients in I. $I.R[[X]] = \{\sum_{finite} x_i f_i ; x_i \in I, f_i \in R[[X]]\}$ it is an ideal of $R[[X]]$. If I is finitely generated then $I[[X]] = I.R[[X]]$.

Remark 3.2.

1. *$\forall P \in Spec(R, S)$, P is S-radical.*
2. *$\forall P \in Spec(R, S)$, $P[[X]] \in Spec(R[[X]], S)$ and then is S-radical of $R[[X]]$.*
3. *$\forall s \in S, I$ ideal of R, $\sqrt[S]{sI} = \sqrt[S]{I}$.*
4. *$Spec(R, \bar{S}) = Spec(R, S)$ and $\forall I$ ideal of R, $\sqrt[\bar{S}]{I} = \sqrt[S]{I}$.*

Proof:

1. It is immediate.

2. Since P is prime disjoint from S then $P[[X]]$ is prime of $R[[X]]$ disjoint from S and then, by 1. it is S-radical of $R[[X]]$.
3. If $I \cap S \neq \emptyset$ then so is sI then $\sqrt[S]{sI} = R = \sqrt[S]{I}$. Otherwise, $\sqrt[S]{sI} = \bigcap_{sI \subseteq P \in Spec(R,S)} P$, here each P is prime not containing s then $sI \subseteq P \Leftrightarrow I \subseteq P$. And then $\sqrt[S]{sI} = \bigcap_{sI \subseteq P \in Spec(R,S)} P = \bigcap_{I \subseteq P \in Spec(R,S)} P = \sqrt[S]{I}$.

Proposition 3.3. *Let $S \subseteq T$ be tow multiplicative sets of the ring R. For all I and J ideals of R, we have the following.*

1. $I \subseteq \sqrt[S]{I}$.
2. $\sqrt{I} = \sqrt[S]{I}, \forall I$ *ideal of R iff* $S \subseteq U(R)$.
3. $I \subseteq J \Rightarrow \sqrt[S]{I} \subseteq \sqrt[S]{J}$
4. $\sqrt[S]{I} \subseteq \sqrt[T]{I}$.
5. $\sqrt[S]{\sqrt[S]{I}} = \sqrt[S]{I}$.
6. $\sqrt[T]{\sqrt[S]{I}} = \sqrt[T]{I}$.
7. $\sqrt[S]{\sqrt[T]{I}} = \sqrt[T]{I}$.
8. $\sqrt[S]{IJ} = \sqrt[S]{I \cap J} = \sqrt[S]{I} \cap \sqrt[S]{J}$. *In particular,* $\sqrt[S]{I^n} = \sqrt[S]{I}$, *for* $n \in \mathbb{N}^*$.

Proof:

1. Immediately from the definition.
2. If $S \subseteq U(R)$ then $Spec(R,S) = Spec(R)$ and then $\sqrt{I} = \sqrt[S]{I}, \forall I$. Conversely, suppose that $S \nsubseteq U(R)$ then $\exists s \in S$ not invertible then there exists $P \in Spec(R)$containing s that is $P \notin Spec(R,S)$. Consequently, $\sqrt[S]{P} = R \neq \sqrt{P} = P$. Contradiction.
3. Immediate from the definition of the S-radical of an ideal.
4. $S \subseteq T$ then $F = \{P \in Spec(R,T); I \subseteq P\} \subseteq G = \{P \in Spec(R,S); I \subseteq P\}$ and then $\bigcap_{P \in G} P \subseteq \bigcap_{P \in F} P$ that is $\sqrt[S]{I} \subseteq \sqrt[T]{I}$.
5. Since $\{P \in Spec(R,S); I \subseteq P\} = \{P \in Spec(R,S); \sqrt[S]{I} \subseteq P\}$ then $\sqrt[S]{\sqrt[S]{I}} = \sqrt[S]{I}$.
6. $I \subseteq \sqrt[S]{I} \subseteq \sqrt[T]{I}$. Then $\sqrt[T]{I} \subseteq \sqrt[T]{\sqrt[S]{I}} \subseteq \sqrt[T]{\sqrt[T]{I}} = \sqrt[T]{I}$ and we get the result.
7. In the same way as the proof of 6.
8. IJ (respectively, $I \cap J) \subseteq P \in spec(R,S) \Rightarrow I$ (or J) $\subseteq P$ then $\sqrt[S]{I} \cap \sqrt[S]{J} = \bigcap_{I or J \subseteq P} P \subseteq \bigcap_{IJ \subseteq P} P = \sqrt[S]{IJ}$, by 3. and 5. we get the inverse inclusion and then $\sqrt[S]{I} \cap \sqrt[S]{J} = \sqrt[S]{IJ}$. In the same way we prove $\sqrt[S]{I} \cap \sqrt[S]{J} = \sqrt[S]{I \cap J}$.

Proposition 3.4.

1. *Each noetherian spectrum ring is S-noetherian spectrum ring.*
2. *Let $S \subseteq T$ be two multiplicative sets of R. If R is an S-noetherian spectrum ring then it is a T-noetherian spectrum ring.*

Proof:

1. Here the ring satisfies (acc) on all ideals in particular, on S-radicals.
2. Let R be a ring satisfying the (acc) on S-radicals and $(\sqrt[T]{I_i})_i$ be an ascending chain of T-radicals of R. Since $\sqrt[T]{I_i} = \sqrt[S]{\sqrt[T]{I_i}}$, the ascending chain $(\sqrt[T]{I_i})_i$ is in fact an ascending chain of S-radicals then it stabilizes.

Proposition 3.5. *If R is an S-SFT ring then each S-radical is an S-radically finite.*

Proof:
Let $I \neq R$ be an S-radical then $I \cap S = \emptyset$. I is S-SFT ideal. Then there exists $s \in S, n \in \mathbb{N}^*, F, J$ ideals of R with J finitely generated such that $sF \subseteq J \subseteq F \subseteq I$ and $x^n \in F, \forall x \in I$. Then $I \subseteq \sqrt{F} \subseteq \sqrt[S]{F} = \sqrt[S]{sF} \subseteq \sqrt[S]{J} \subseteq \sqrt[S]{F} \subseteq \sqrt[S]{I}$. We take the $\sqrt[S]{\ }$ of all the terms of the previous inclusions and we get $\sqrt[S]{I} \subseteq \sqrt[S]{F} = \sqrt[S]{J} \subseteq \sqrt[S]{F} \subseteq \sqrt[S]{I}$ then $\sqrt[S]{I} = \sqrt[S]{J}$.

Theorem 3.6. *For R a ring and S a multiplicative set of R, the following are equivalent:*

1. *R has (acc) on S-radical ideals (i. e. R has S-noetherian spectrum).*
2. *Each S-radical ideal is the S-radically finite.*
3. *Each prime ideal disjoint from S is the S-radically of a finite.*
4. *R has (acc) on prime ideals disjoint from $S$and $min(I, S)$ is finite for each ideal I disjoint from S.*
5. *R has (acc) on prime ideals disjoint from S and $min(I, S)$ is finite for each finitely generated ideal I disjoint from S.*
   Here $min(I, S)$ denotes the set of primes minimal on I and disjoint from S.

Proof: 2. $\Rightarrow$ 1. Let $(I_k, k \leq 1)$ be an ascending chain of S-radicals and $I = \cup_k I_k$. $\sqrt[S]{I}$ is an S-radical then there exists a finitely generated ideal $F \subseteq I$ such that $\sqrt[S]{I} = \sqrt[S]{F}$. Then there exists $l \geq 1$ such that $F \subseteq I_l$ and then $I \subseteq \sqrt[S]{I} = \sqrt[S]{F} \subseteq \sqrt[S]{I_l} = I_l \subseteq I$ then the chain $(I_k, k \leq 1)$ stabilizes.

3. $\Rightarrow$ 2. Suppose that the set F of all S-radicals that are not S-radically finite is not empty. This set ordered by inclusion is inductive. Let I be a maximal element of it. We prove that I is prime. Suppose that it is not prime, then there exist $x_1 x_2 \in I, x_i \notin I$ then $I \subsetneq \sqrt[S]{x_i R + I}$ then there exists $H_i \subseteq x_i R + I$ finitely generated such that $\sqrt[S]{x_i R + I} = \sqrt[S]{H_i}$. We have $I^2 \subseteq (x_1 R + I)(x_2 R + I) \subseteq I$ then $\sqrt[S]{I^2} \subseteq \sqrt[S]{(x_1 R + I)(x_2 R + I)} \subseteq \sqrt[S]{I}$. By Proposition 3.3- 8., $\sqrt[S]{I} \subseteq \sqrt[S]{(x_1 R + I)(x_2 R + I)} =\subseteq \sqrt[S]{(x_1 R + I)} \cap \subseteq \sqrt[S]{(x_2 R + I)} = \sqrt[S]{H_1}) \cap \subseteq \sqrt[S]{H_2} = \sqrt[S]{H_1 H_2}) \subseteq \sqrt[S]{I}$ then $I = \sqrt[S]{I} = \sqrt[S]{H_1 H_2})$. This is a contradiction because I is not S-radically finite.

Then I is prime. By 3., $I \notin F$, contradiction, then $F = \emptyset$ and then all S-radicals are S-radically finite.

1. $\Rightarrow$ 4.: Prime ideals disjoint from S are S-radicals then the (acc) on S-radicals implies the (acc) on prime ideals disjoint from S. Suppose that there exists an ideal I with $min(I, S)$ is infinite. Since $min(I, S) = min(\sqrt[S]{I}, S)$, we can assume that I is an S-radical. By 1. the set of all S-radicals I with infinite $min(I, S)$ has maximal elements. Take I a maximal element of the previous set, I is not prime (if it is prime $min(I, S) = \{I\}$ finite) then there exist $x_1, x_2 \in R$ such that $x_i \notin I, x_1 x_2 \in I$ we have $I \subsetneq \sqrt[S]{I + x_i R}$ then $min(I + x_i R, S)$ is finite. By , $I = \sqrt[S]{I} = \sqrt[S]{I^2} \subseteq \sqrt[S]{(I + x_1 R)(I + x_2 R)} = \sqrt[S]{(I + x_1 R)} \cap \sqrt[S]{(I + x_2 R)} \subseteq \sqrt[S]{I}$ then $\bigcap_{p \in min(I,S)} P = \bigcap_{1 \leq k \leq n} P_k$,with

$\{P_1, ..., P_n\} = min(I + x_1R, S) \cup min(I + x_2R, S)$. $\forall P, \bigcap\limits_{1\leq k\leq n} P_k \subseteq P$ then one of the $P_k^{'}s$ is contained in P since $I \subseteq P_k$ and P is minimal over S then $P = P_k$ and then $min(I, S)$ is finite, contradiction.

4. $\Rightarrow$ 5. is clear.
5. $\Rightarrow$ 3.: Suppose that there exists a prime P disjoint from S that is not S-radically finite, then $P \neq \{0\}$. Take $x \in P$, by 5. $min(xR, S)$ is finite. Suppose $\forall Q \in min(xR, S), P \subseteq Q$ then $P \subseteq \bigcap\limits_{Q\in min(xR,S)} Q = \sqrt[S]{xR} \subseteq \sqrt[S]{P} = P$ this contradicts the fact that P is not S-radically finite. Then there exists $Q \in min(xR, S)$ such that $P \nsubseteq Q$. Take $x_Q \in P \setminus Q$ and I_1 the ideal generated by the finite set $\{x, x_Q; P \nsubseteq Q \in min(xR, S)\}$. Since P is not S-radically finite then $I_1 \subsetneq P$. I_1 is finitely generated then $min(I_1, S)$ is finite. As for $min(xR, S)$ there exists $Q_1 \in min(I_1, S)/ P \nsubseteq Q_1$. $\sqrt[S]{xr} = \bigcap\limits_{Q\in min(xR,S)} Q \subseteq \sqrt[S]{I_1} \subseteq Q_1$ then there exists $Q \in min(xR, S)$ / $Q \subseteq Q_1$. $P \nsubseteq Q_1$ then $P \nsubseteq Q$. Since $x_Q \in Q_1 \setminus Q$ then $Q \subsetneq Q_1$.

Take $x_{Q_1} \in P \setminus Q_1$ and I_2 the ideal generated by the set $\{x, x_Q; P \nsubseteq Q \in min(xR, S), x_{Q_1}; P \nsubseteq Q_1 \in min(I_1, S)\}$. In the same way we find $Q_2 \in min(I_2, S)$; $P \subseteq Q_2$ and a chain $Q \subsetneq Q_1 \subsetneq Q_2$. Repeating the previous process, we construct a chain of finitely generated ideals contained in P $xR \subseteq I_1 \subseteq I_2 \subseteq ... \subseteq I_k...$ and a chain of primes $Q \subsetneq Q_1 \subsetneq Q_2 \subsetneq ....$ R has (acc) on primes disjoint from S then the previous process must stop at a level k that is for all $Q_k \in min(I_k, S), P \subseteq Q_k$ and then $P \subseteq \bigcap\limits_{Q_k\in min(I_k,S)} = \sqrt[S]{I_k} \subseteq \sqrt[S]{P} = P$ and then $P = \sqrt[S]{I_k}$. This contradicts the supposition that P is not S-radically finite.

Theorem 3.7. *An S-SFT ring is an S-notherian spectrum ring.*

Proof: Immediatly by Proposition 3.5 and Theorem 3.6.

Theorem 3.8. *Let R be a ring and S be a multiplicative set of R. For the following conditions:*

1. *There exists an ideal I of R disjoined to S such that $I[[X]] \nsubseteq \sqrt[S]{I.R[[X]]}$.*
2. *There exists an ideal $P \in Spec(R, S)$ such that $P[[X]] \neq \sqrt[S]{P.R[[X]]}$.*
3. *The ring R is not a S-SFT ring,*
   we have 1) and 2) are equivalent and imply 3).

Proof:
1.$\Rightarrow$ 2. : $I[[X]] \nsubseteq \sqrt[S]{I.R[[X]]}$ then there exists $P \in spec(R[[X]], S)$ such that $I.R[[X]] \subseteq P \nsupseteq I[[X]]$. Let $\mathcal{P} = R \cap P$, then $\sqrt[S]{\mathcal{P}.R[[X]]} \subseteq P$ and $I[[X]] \nsubseteq \sqrt[S]{\mathcal{P}.R[[X]]}$. We have $I.R[[X]] \subseteq P$ then $I \subseteq \mathcal{P}$ and then $I[[X]] \subseteq \mathcal{P}[[X]]$. Consequently, $\mathcal{P}[[X]] \neq \sqrt[S]{\mathcal{P}.R[[X]]}$.
2.$\Rightarrow$ 1. is immediate.
2.$\Rightarrow$ 3.

Let $P \in Spec(R, S)$ such that $P[[X]] \neq \sqrt[S]{P.R[[X]]}$. Suppose R is an S-SFT ring then P is an S-SFT ideal then there exists $s \in S, n \in \mathbb{N}, B \subseteq P$ finitely generated such that $\forall x \in P, sx^n \in B \Rightarrow (sx)^n \in B$ and then $(sx)^n = 0 \in R/B$. By [7] Lemma 4 of Arnold, $sP[[X]] \subseteq \sqrt{B[[X]]} \subseteq \sqrt{P[[X]]}$. Then $\sqrt[S]{sP[[X]]} \subseteq \sqrt[S]{\sqrt{B[[X]]}} \subseteq \sqrt[S]{\sqrt{P[[X]]}}$.

Since $\sqrt[S]{sP[[X]]} = \sqrt[S]{P[[X]]} = P[[X]]$ and $\sqrt[S]{\sqrt{B[[X]]}} = \sqrt[S]{B[[X]]}$, then

$P[[X]] \subseteq \sqrt[S]{B[[X]]} \subseteq P[[X]]$, that is $\sqrt[S]{B[[X]]} = P[[X]]$. On the other hand, B is finitely generated then $B[[X]] = B.R[[X]] \subseteq P.R[[X]] \subseteq P[[X]]$ then $P[[X]] = \sqrt[S]{B[[X]]} \subseteq \sqrt[S]{P.R[[X]]} \subseteq P[[X]]$, then $\sqrt[S]{P.R[[X]]} = P[[X]]$, contradiction. Consequently, R is not a S-SFT ring.

Corollary 3.9. *If R is an S-SFT ring then: $\sqrt[S]{I[[X]]} = \sqrt[S]{I.R[[X]]}$, $\forall I$ ideal of R. In particular if I is prime, $I[[X]] = \sqrt[S]{I.R[[X]]}$*

4 Conclusion

We introduced two types of rings: the S-sft ring and the S-noetherian spectrum ring. We characterize and illustrate them by many examples. We generalize some known results.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Esmaeelnezhad A, et al. On S-semistar-noetherian domains. International Electronical Journal of Algebra. 2015;18:57-71.

[2] Hamed A, et al. S-noetherian rings of the forms $\mathcal{A}[\mathcal{X}]$ and $\mathcal{A}[[\mathcal{X}]]$. Communications in Algebra. 2015;43(9):3848-3856.

[3] Kim H, et al. On S-strong Mori domains. J. Algebra. 2014;416:314-332.

[4] Lim JW. A note on S-Noetherian domains. Kyungpook Math. J. 2015;55:507-514.

[5] Liu Z. On S-Noetherian rings. Arch. Math. 2007;43:55-60.

[6] Anderson DD, Dumitrescu T. S-Noetherian rings. Communications in Algebra. 2002;30(9): 44074416.

[7] Arnold JT. Krull dimension in power series rings. Trans. Amer. Math. Soc. 1973;177.

[8] Arnold JT. Power series rings over Prufer domains. Pacific J. of Mathematics. 1973;44(1).

[9] Arnold JT, Brewer JW. On flat overrings, ideal transforms and generalized transforms of a commutative ring. J. of Algebra. 1971;18:254-263.

[10] Rush D. Noetherian spectrum on rings and modules. Glasgow. Math. J. 2011;53:707-715.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sciencedomain.org/review-history/25090