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On the Solution of Linear Differential Equations with Polynomial Coefficients near the Origin and Infinity

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

A linear differential equation with polynomial coefficients is studied. In the preceding study given in J. Adv. Math. Comput. Sci. 2018; 28 (3) 1-15, the equation is expressed in terms of blocks of classified terms, and the full solutions near the origin are presented for the differential equations of the second order and with two blocks of classified terms. In the present study, it is shown that the solutions near infinity are easily obtained and the asymptotic behaviors are discussed with the aid of a theorem given in the preceding paper.

Keywords: Linear differential equations with polynomial coefficients; blocks of classified terms; fractional calculus; change of variables.

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1 Introduction

In the preceding paper [1], differential equations of order $l_x \in \mathbb{Z}_{>0}$, with coefficients of polynomials, are studied. They take the form:

$$\sum_{k=0}^{l_x} \sum_{m=0}^{\infty} a_{k,m} t^m \frac{d^k}{dt^k} u(t) = \sum_{k=0}^{l_x} (a_{k,0} + a_{k,1} \cdot t + a_{k,2} \cdot t^2 + a_{k,3} \cdot t^3 + \cdots) \cdot \frac{d^k}{dt^k} u(t) = 0, \quad t > 0, \tag{1.1}$$

where $a_{k,m}$ for $k \in \mathbb{Z}_{[0,l_x]}$ and $m \in \mathbb{Z}_{>-1}$ are constants. It was assumed that a finite number of the constants are nonzero.

Here $\mathbb{R}$ and $\mathbb{Z}$ are the sets of all real numbers and all integers, respectively, and $\mathbb{Z}_{[a,b]} = \{n \in \mathbb{Z} | a \leq n \leq b\}$ for $a, b \in \mathbb{Z}$ satisfying $a < b$. We also use $\mathbb{C}$ which is the set of all complex numbers, and $\mathbb{Z}_{>a} = \{n \in \mathbb{Z} | n > a\}$, $\mathbb{Z}_{<a} = \{n \in \mathbb{Z} | n < a\}$ for $a \in \mathbb{Z}$, and $\mathbb{R}_{>a} = \{x \in \mathbb{R} | x > a\}$ for $a \in \mathbb{R}$. We use $(z)_k$ and $(z)_k^-$ for $z \in \mathbb{C}$, $k \in \mathbb{Z}_{>-1}$, which denote $(z)_k = \prod_{m=0}^{k-1} (z+m)$ if $k \in \mathbb{Z}_{>0}$, and $(z)_0 = 1$, as usual, and

$$(z)_k^- = \prod_{m=0}^{k-1} (z - m) = (-1)^k (-z)_k, \quad k \in \mathbb{Z}_{>0}, \tag{1.2}$$

and $(z)_0^- = 1$.

In [1], the terms of Equation (1.1) are reassembled as

$$\sum_{l=-\infty}^{l_x} D_t^l u(t) = 0, \quad t > 0, \tag{1.3}$$

where

$$D_t^l u(t) = \sum_{k=\max\{0,l\}}^{l_x} a_{k,k-l} \cdot t^{k-l} \frac{d^k}{dt^k} u(t), \tag{1.4}$$

each of $D_t^l u(t)$ is called a block of classified terms.

When $l_x = 2$, Equation (1.3) is expressed as

$$D_t^2 u(t) + D_t^1 u(t) + D_t^0 u(t) + D_t^{-1} u(t) + D_t^{-2} u(t) + \cdots = 0, \quad t > 0, \tag{1.5}$$

where

$$D_t^2 = a_{2,0} \frac{d^2}{dt^2}, \quad D_t^1 = a_{2,1} t \cdot \frac{d^2}{dt^2} + a_{1,0} \frac{d}{dt}, \quad D_t^0 = a_{2,2} t^2 \cdot \frac{d^2}{dt^2} + a_{1,1} t \cdot \frac{d}{dt} + a_{0,0},$$
$$D_t^{-1} = a_{2,3} t^3 \cdot \frac{d^2}{dt^2} + a_{1,2} t^2 \cdot \frac{d}{dt} + a_{0,1} t, \quad D_t^{-2} = a_{2,4} t^4 \cdot \frac{d^2}{dt^2} + a_{1,3} t^3 \cdot \frac{d}{dt} + a_{0,2} t^2, \quad \cdots . \tag{1.6}$$

When D_t^l is operated on t^α for $\alpha \in \mathbb{C} \backslash \mathbb{Z}$, we have

$$D_t^l t^\alpha = A_l(\alpha) t^{\alpha - l}, \tag{1.7}$$

where

$$A_l(\alpha) = \sum_{k=\max\{0,l\}}^{l_x} a_{k,k-l} \cdot (\alpha)_k^- . \tag{1.8}$$

When we discuss a differential equation of order l_x, the following condition is adopted.

Condition 1.1. We consider such a differential equation is not regarded as a differential equation of $u'(t)$, so that $\sum_{m=0}^{\infty} |a_{l_x,m}| \neq 0$ and $\sum_{m=0}^{\infty} |a_{0,m}| \neq 0$.

In [1], special attention is focussed on Equation (1.5) for the case in which there exist two nonzero blocks of classified terms, so that the equation is expressed as

$$D_t^l u(t) + D_t^{l-m} u(t) = 0, \quad m \in \mathbb{Z}_{>0}, \tag{1.9}$$

Remark 1.1. By (1.6) for $l_x = 2$, we see that Equation (1.9) for $l = -1, -2, \dots$ are equivalent to the one for $l = 0$, and the differential equation for $l = 1$ is equivalent to the one for $l = 0$ when $a_{0,0} = 0$. We note that the differential equation for $l = 2$ is equivalent to a special one for $l = 0$. Hence we study only the differential equation for $l = 0$.

In [1], special attention is focussed on the solutions of

$$D_t^0 u(t) + D_t^{-1} u(t) = 0, \tag{1.10}$$

In [2, 3, 4], the solutions of Kummer's and the hypergeometric differential equation, which are special ones of Equation (1.10), were studied with the aid of distribution theory, and of the AC-Laplace transform, that is the Laplace transform supplemented by its analytic continuation. In the study, the following condition was adopted.

Condition 1.2. $u(t)$ is expressed as a linear combination of $g_\nu(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1}$ for $t > 0$ and $\nu \in S$, where S is a set of $\nu \in \mathbb{R}_{>-M} \backslash \mathbb{Z}_{<1}$ for some $M \in \mathbb{Z}_{>-1}$.

We then express $u(t)$ as follows:

$$u(t) = \sum_{\nu \in S} u_{\nu-1} \frac{1}{\Gamma(\nu)} t^{\nu-1}, \tag{1.11}$$

where $u_{\nu-1} \in \mathbb{C}$ are constants. Because of this condition, we obtained the solutions which are expressed by a power series of t multiplied by a power t^α:

$$u(t) = t^\alpha \sum_{k=0}^{\infty} p_k t^k, \tag{1.12}$$

where $\alpha \in \mathbb{C} \backslash \mathbb{Z}_{<0}$, $p_k \in \mathbb{C}$ and $p_0 \neq 0$.

In [1], for Equation (1.10), the solutions in the form of (1.12) are shown to be given by the generalized hypergeometric function. When $l_x = 2$, the solutions of Equation (1.10), which are of the form (1.12), are expressed by

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} z^k, \quad {}_2F_0(a, b; ; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!} z^k, \tag{1.13}$$

$$ {}_1F_1(a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{k!(c)_k} z^k, \quad {}_0F_1(; c; z) = \sum_{k=0}^{\infty} \frac{1}{k!(c)_k} z^k. \tag{1.14}$$

The first series in (1.13) and (1.14) are the hypergeometric and the confluent hypergeometric series, respectively. In Section 2, we reproduce the theorem giving the results for the solution of Equation (1.10) for $l_x = 2$, with a correction. In [1], every equation of Equation (1.9) for $m \in \mathbb{Z}_{>1}$ is shown to reduced to a differential equation of the form of (1.10) by a change of variable.

In [5], the asymptotic behaviors as $t \to \infty$ are discussed for the confluent hypergeometric function, which is a solution of Kummer's differential equation, in the standpoint of fractional calculus.

It is the purpose of the present paper to show how the behaviors near infinity are obtained for all the solutions given in the above-mentioned theorem. In Section 3, we first show that the solutions near infinity, which take the form:

$$u(t) = t^{\beta} \sum_{k=0}^{\infty} q_k t^{-k}, \tag{1.15}$$

are easily obtained with the aid of the theorem. Discussion is then given on the behaviors as $t \to \infty$, for the solutions given in the theorem. As a by-product, we obtain the solution of the following equation in Section 3.2:

$$t^3 \frac{d^2 u}{dt^2} - u = 0. \tag{1.16}$$

In Section 3.3, we use a formula for the confluent hypergeometric function, which is given in [5]. In Section 4, discussion based on fractional calculus is given on the derivation of the corresponding formula for the hypergeometric function.

2 Solution of Equation (1.10) for $l_x = 2$ and $l = 0$

From now on, we restrict the discussion to the case of $l_x = 2$.

We introduce notation ${}_n\tilde{D}_t^l$ which represents D_t^l, when the coefficient of t^n is nonzero and those of t^m for $m > n$ are all zero. The differential equations belonging to Equation (1.10) for $l = 0$ are classified into

$$ {}_2\tilde{D}_t^0 u(t) + {}_n\tilde{D}_t^{-1} u(t) = 0, \quad n = 3, 2, 1, \tag{2.1}$$

$$ {}_m\tilde{D}_t^0 u(t) + {}_3\tilde{D}_t^{-1} u(t) = 0. \quad m = 2, 1, 0. \tag{2.2}$$

We call Equation (2.1) for $n \in \mathbb{Z}_{[1,3]}$ as (2.1-n), and Equation (2.2) for $m \in \mathbb{Z}_{[0,2]}$ as (2.2-m), where (2.1-3) and (2.2-2) represent the same equation.

We use a, b and c, which satisfy $a_{1,1} = a_{2,2}(1 + a + b)$ and $a_{0,0} = a_{2,2} \cdot ab$ when $a_{2,2} \neq 0$, and $a_{0,0} = a_{1,1} \cdot c$ when $a_{2,2} = 0$ and $a_{1,1} \neq 0$. Using these in (1.6), we obtain

$$ {}_2\tilde{D}_t^0 = a_{2,2}[t^2 \cdot \frac{d^2}{dt^2} + (1 + a + b)t \cdot \frac{d}{dt} + ab], \quad {}_1\tilde{D}_t^0 = a_{1,1}(t \cdot \frac{d}{dt} + c), \quad {}_0\tilde{D}_t^0 = a_{0,0}. \tag{2.3}$$

When $a_{0,0} = 0$, we put $b = 0$ and $c = 0$ in (2.3). We use $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$, which satisfy $a_{1,2} = a_{2,3}(1+\tilde{a}+\tilde{b})$ and $a_{0,1} = a_{2,3} \cdot \tilde{a}\tilde{b}$ when $a_{2,3} \neq 0$, and $a_{0,1} = a_{1,2} \cdot \tilde{c}$ when $a_{2,3} = 0$ and $a_{1,2} \neq 0$. Using these in (1.6), we obtain

$$ {}_3\tilde{D}_t^{-1} = a_{2,3} \cdot t[t^2 \cdot \frac{d^2}{dt^2} + (1 + \tilde{a} + \tilde{b})t \cdot \frac{d}{dt} + \tilde{a}\tilde{b}], \quad {}_2\tilde{D}_t^{-1} = a_{1,2} \cdot t(t \cdot \frac{d}{dt} + \tilde{c}),$$
$$ {}_1\tilde{D}_t^{-1} = a_{0,1} \cdot t. \tag{2.4}$$

When $a_{0,1} = 0$, we put $\tilde{b} = 0$ and $\tilde{c} = 0$ in the first two equations of (2.4).

Remark 2.1. In Section 7.21 of [6], terminologies "regular singular point" and "irregular singular point" are used. For an equation, which is expressed by

$$[(t-c)^2 \frac{d^2}{dt^2} + (t-c)p(t)\frac{d}{dt} + q(t)]u(t) = 0, \tag{2.5}$$

point $t = c$ is called a singular point, if $\frac{p(t)}{t-c}$ or $\frac{q(t)}{(t-c)^2}$ is not analytic at c. If the point $t = c$ is a singular point, it is said to be regular or irregular at $t = c$, according as both $p(t)$ and $q(t)$ are analytic at $t = c$, or not so. By this terminology, the point $t = 0$ is a regular singular point of Equation (2.1), and it is an irregular singular point of Equations (2.2-1) and (2.2-0). We note that there exist two, one and no solutions of the form (1.12) for Equations (2.1) satisfying $a - b \notin \mathbb{Z}$, (2.2-1) and (2.2-0), respectively.

The differential equation (2.1-3) for $a_{0,0} = 0$ is the hypergeometric differential equation, whose solutions are the hypergeometric functions. The differential equation (2.1-2) for $a_{0,0} = 0$ is Kummer's differential equation, whose solutions are the confluent hypergeometric functions. Laguerre's differential equation is a special one of Kummer's differential equation; See Chapter VIII in [7], and Chapter 13 in [8].
The following theorem is Theorem 2.2 in [1], with minor corrections.

Theorem 2.1. We have the following solutions of the form (1.12) for Equations (2.1)∼(2.2).

(i). If $a_{0,0} \neq 0$, $a_{0,1} \neq 0$ and $a - b \notin \mathbb{Z}$, we have the pairs of solutions of (2.1-3), (2.1-2) and (2.1-1), respectively, which are given by

$$\phi_\alpha(t) = t^\alpha \cdot {}_2F_1(\tilde{a} + \alpha, \tilde{b} + \alpha; 1 + a + b + 2\alpha; -\frac{a_{2,3}}{a_{2,2}}t), \tag{2.6}$$

$$\phi_\alpha(t) = t^\alpha \cdot {}_1F_1(\tilde{c} + \alpha; 1 + a + b + 2\alpha; -\frac{a_{1,2}}{a_{2,2}}t), \tag{2.7}$$

$$\phi_\alpha(t) = t^\alpha \cdot {}_0F_1(; 1 + a + b + 2\alpha; -\frac{a_{0,1}}{a_{2,2}}t), \tag{2.8}$$

for $\alpha = -a$ and $\alpha = -b$.

(ii). If $a_{0,0} = 0$, $a_{0,1} \neq 0$ and $-a \notin \mathbb{Z}$, pairs of solutions of (2.1-3), (2.1-2) and (2.1-1) are given by

$$\phi_0(t) = {}_2F_1(\tilde{a}, \tilde{b}; 1 + a; -\frac{a_{2,3}}{a_{2,2}}t), \quad \phi_{-a}(t) = t^{-a} \cdot {}_2F_1(\tilde{a} - a, \tilde{b} - a; 1 - a; -\frac{a_{2,3}}{a_{2,2}}t); \tag{2.9}$$

$$\phi_0(t) = {}_1F_1(\tilde{c}; 1 + a; -\frac{a_{1,2}}{a_{2,2}}t), \quad \phi_{-a}(t) = t^{-a} \cdot {}_1F_1(\tilde{c} - a; 1 - a; -\frac{a_{1,2}}{a_{2,2}}t); \tag{2.10}$$

$$\phi_0(t) = {}_0F_1(; 1 + a; -\frac{a_{0,1}}{a_{2,2}}t), \quad \phi_{-a}(t) = t^{-a} \cdot {}_0F_1(; 1 - a; -\frac{a_{0,1}}{a_{2,2}}t), \tag{2.11}$$

respectively, which are (2.6)∼(2.8) for $b = 0$.

(iii). If $a_{0,0} \neq 0$, $a_{0,1} = 0$ and $a - b \notin \mathbb{Z}$, pairs of solutions of (2.1-3) and (2.1-2) are given by (2.6) for $\tilde{b} = 0$, and by (2.7) for $\tilde{c} = 0$, respectively.

(iv). If $a_{0,0} \neq 0$ and $a_{0,1} \neq 0$, we have only one solution of (2.2-1) given by

$$\psi_{-c}(t) = t^{-c} \cdot {}_2F_0(\tilde{a} - c, \tilde{b} - c; ; -\frac{a_{2,3}}{a_{1,1}}t). \tag{2.12}$$

The second factor on the righthand side of (2.12) is a polynomial, when $\tilde{a} - c \in \mathbb{Z}_{<1}$ or $\tilde{b} - c \in \mathbb{Z}_{<1}$. If such is not the case, it is an infinite series which has zero radius of convergence.

(v). If $a_{0,0} = 0$ and $a_{0,1} \neq 0$, only one solution of (2.2-1) is given by (2.12) for $c = 0$.

(vi). If $a_{0,0} \neq 0$ and $a_{0,1} = 0$, only one solution of (2.2-1) is given by (2.12) for $\tilde{b} = 0$.

(vii). There exists no solution of the form (1.12) for (2.2-0).

Remark 2.2. In Theorem 2.2 of [1], the case (vii) is not included, since there exists no solution of the form (1.12) in this case.

Remark 2.3. In [1], the first equation in (2.11) is given by $\phi_0(t) = {}_0F_1(;1+a;-\frac{a_{1,1}}{a_{2,2}}t)$, which is in error and should be corrected as in (2.11).

3 Behaviors of the Solutions Given in Theorem 2.1 as $t \to \infty$ and $t \to 0$

Lemma 3.1. *We put* $x = \frac{1}{t}$, $v(x) = u(t)$, *and*

$$ {}_m\tilde{D}_x^0 v(x) := \frac{1}{t} \cdot {}_{m+1}D_t^{-1} u(t), \quad m = 2, 1, 0, \tag{3.1}$$

$$ {}_n\tilde{D}_x^{-1} v(x) := x \cdot {}_{n-1}D_t^0 u(t), \quad n = 3, 2, 1. \tag{3.2}$$

Then

$$ {}_2\tilde{D}_x^0 v(x) + {}_n\tilde{D}_x^{-1} v(x) = 0, \quad n = 3, 2, 1, \tag{3.3}$$

$$ {}_m\tilde{D}_x^0 v(x) + {}_3\tilde{D}_x^{-1} v(x) = 0, \quad m = 2, 1, 0, \tag{3.4}$$

where

$$ {}_2\tilde{D}_x^0 = a_{2,3}[x^2 \cdot \frac{d^2}{dx^2} + (1 - \tilde{a} - \tilde{b})x \cdot \frac{d}{dx} + \tilde{a}\tilde{b}], \quad {}_1\tilde{D}_x^0 = a_{1,2}(-x \cdot \frac{d}{dx} + \tilde{c}), \quad {}_0\tilde{D}_x^0 = a_{0,1}, \tag{3.5}$$

in place of (2.3)*, and*

$$ {}_3\tilde{D}_x^{-1} = a_{2,2} \cdot x[x^2 \cdot \frac{d^2}{dx^2} + (1 - a - b)x \cdot \frac{d}{dx} + ab], \quad {}_2\tilde{D}_x^{-1} = a_{1,1} \cdot x(-x \cdot \frac{d}{dx} + c),$$
$$ {}_1\tilde{D}_x^{-1} = a_{0,0} \cdot x, \tag{3.6}$$

in place of (2.4)*. We call Equation* (3.3) *for* $n \in \mathbb{Z}_{[1,3]}$ *as* (3.3-n)*, and Equation* (3.4) *for* $m \in \mathbb{Z}_{[0,2]}$ *as* (3.4-m)*, where* (3.3-3) *and* (3.4-2) *represent the same equation.*

Lemma 3.2. *When Equations* (2.1-3)*,* (2.1-2)*,* (2.1-1)*,* (2.2-1) *and* (2.2-0) *hold, Equations* (3.4-2)*,* (3.4-1)*,* (3.4-0)*,* (3.3-2) *and* (3.3-1)*, respectively, hold.*

Remark 3.1. The solutions of Equations (3.3-n) and (3.4-m), respectively, for $n \in \mathbb{Z}_{[1,3]}$ and $m \in \mathbb{Z}_{[0,2]}$, are given by those of Equations (2.1-n) and (2.2-m) with $a_{2,2}$, $a_{1,1}$, $a_{0,0}$, a, b, c, ϕ, ψ and t exchanged by $a_{2,3}$, $-a_{1,2}$, $a_{0,1}$, $-\tilde{a}$, $-\tilde{b}$, $-\tilde{c}$, $\tilde{\phi}$, $\tilde{\psi}$ and x, respectively.

Remark 3.2. As a consequence of Remark 3.1, from the solutions of the form (1.12) for Equations (2.1) and (2.2) given in Theorem 2.1, we obtain the solutions of the form:

$$ v(x) = t^{\alpha} \sum_{k=0}^{\infty} p_k x^k, \tag{3.7}$$

for Equations (3.3) and (3.4), where $\alpha \in \mathbb{C} \backslash \mathbb{Z}_{<0}$, $p_k \in \mathbb{C}$ and $p_0 \neq 0$.

Remark 3.3. In Section 7.22 in [6], it is stated that an equation belonging to (2.1)$\sim$(2.2) is said to have a regular or irregular singular point at $t = \infty$, according as the equation associated to it in Lemma 3.2 has a regular or irregular singular point at the point $x = 0$. By Remak 2.1, the equations in (3.3) and Equations (3.4-1) and (3.4-0) have a regular and an irregular singular point, respectively, at $x = 0$. As a consequence, by Lemma 3.2, Equations (2.1-3), (2.2-1) and (2.2-0) have a regular singular point at $x = \infty$, and Equations (2.1-2) and (2.1-1) have an irregular singular point at $x = \infty$.

3.1 Solutions of (2.1-1) with $a_{0,1} \neq 0$ for $|z| \to \infty$

When Equation (2.1-1) with $a_{0,1} \neq 0$ holds, we have the pair of solutions given in (2.8). In this case, Lemma 3.2 shows that Equation (3.4-0) with $a_{0,1} \neq 0$ holds, and Remark 3.1 shows that Equation (3.4-0) has no solution, since Equation (2.2-0) has no solution as shown in Theorem 2.1(vii).

We put $a_{2,2} = 1$, $a_{0,1} = -\delta$, $z = t$, $\zeta = z^{-1}$, $w(z) := u(t)$ and $\tilde{w}(\zeta) := v(x)$. Then Equations (2.1-1) and (3.4-0) are expressed by

$$z^2 \cdot \frac{d^2 w}{dz^2} + (1+a+b)z \cdot \frac{dw}{dz} + abw - \delta z w = 0, \tag{3.8}$$

$$-\delta \tilde{w} + \zeta^3 \cdot \frac{d^2 \tilde{w}}{d\zeta^2} + (1-a-b)\zeta^2 \cdot \frac{d\tilde{w}}{d\zeta} + ab\zeta\tilde{w} = 0, \tag{3.9}$$

and if $a - b \notin \mathbb{Z}$, we obtain the following solutions of (3.8), with the aid of (2.8):

$$\Phi_{-a}(z) := \delta^{-a} \cdot \phi_{-a}(t) = (\delta z)^{-a} \cdot {}_0F_1(;1-a+b;\delta z), \tag{3.10}$$

$$\Phi_{-b}(z) := \delta^{-b} \cdot \phi_{-b}(t) = (\delta z)^{-b} \cdot {}_0F_1(;1+a-b;\delta z). \tag{3.11}$$

Remark 3.4. When $n \in \mathbb{Z}_{>-1}$, $a = b + n - \epsilon$ and $\delta = 1$, by using (3.11) and (3.10), respectively, we obtain two solutions of (3.8):

$$z^{-b} \cdot {}_0F_1(;1+n-\epsilon;z)$$
$$= z^{-b}[1 + \sum_{k=1}^{\infty} \frac{1}{k!(1+n-\epsilon)(1+n-\epsilon+1)(1+n-\epsilon+2)\cdots(1+n-\epsilon+k-1)} z^k], \tag{3.12}$$

$$\epsilon \cdot n!(1-n+\epsilon)_{n-1} \cdot z^{-b-n+\epsilon} \cdot {}_0F_1(;1-n+\epsilon;z) = (1-\delta_{n,0})\epsilon \cdot n! \cdot z^{-b-n+\epsilon} \sum_{k=0}^{n-1} \frac{(1-n+\epsilon)_{n-1}}{k!(1-n+\epsilon)_k} z^k$$
$$+ z^{-b+\epsilon}[1 + \sum_{k'=1}^{\infty} \frac{n!}{(n+k')!(\epsilon+1)(\epsilon+2)\cdots(\epsilon+k')} z^{k'}], \tag{3.13}$$

both of which tend to $z^{-b} \cdot {}_0F_1(;1+n;z)$ as $\epsilon \to 0$, where $\delta_{n,0}$ denotes Kronecker's delta, which is equal to 1, if $n = 0$, and to 0, if otherwise.

Lemma 3.3. *When $n \in \mathbb{Z}_{>-1}$, $a = b + n$ and $\delta = 1$, we have two solutions of* (3.8)*:*

$$w_1(z) := z^{-b} \cdot {}_0F_1(;1+n;z), \tag{3.14}$$

$$w_2(z) := (1-\delta_{n,0})n! \cdot z^{-b-n} \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}(n-k-1)!}{k!} z^k$$
$$+ z^{-b} \cdot {}_0F_1(;1+n;z) \log z - z^{-b} \sum_{k=1}^{\infty} \frac{n!(\phi(n+k) - \phi(n) + \phi(k))}{k!(n+k)!} z^k, \tag{3.15}$$

where

$$\phi(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}. \tag{3.16}$$

Proof. To obtain (3.15), we use (3.13) and (3.12) in

$$w_2(z) = \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon}[\epsilon \cdot n!(1-n+\epsilon)_{n-1} \cdot z^{-b-n+\epsilon} \cdot {}_0F_1(;1-n+\epsilon;z) - z^{-b} \cdot {}_0F_1(;1+n-\epsilon;z)]. \tag{3.17}$$

□

Putting $b = 0$ and $n = 1$ in this lemma, we obtain the following lemma.

Lemma 3.4. *When $a = 1$, $b = 0$ and $\delta = 1$, we have two solutions of* (3.8)*:*

$$w_1(z) := {}_0F_1(;2;z), \tag{3.18}$$

$$w_2(z) := z^{-1} + {}_0F_1(;2;z)\log z - \sum_{k=1}^{\infty} \frac{\phi(k) + \phi(k+1) - 1}{k!(k+1)!} z^k, \tag{3.19}$$

where $\phi(k)$ is given by (3.16)*.*

3.2 Solutions of (2.2-0) with $a_{0,0} \neq 0$ for $|z| \to \infty$

When (2.2-0) with $a_{0,0} \neq 0$ holds, Theorem 2.1(vii) shows that there exists no solution of the form (1.12). In this case, Lemma 3.2 shows that (3.3-1) with $a_{0,0} \neq 0$ holds, and Remark 3.1 shows that (2.1-1) and (2.8), with $a_{2,2}$, $a_{0,0}$, $a_{0,1}$, a, b, ϕ and t replaced by $a_{2,3}$, $a_{0,1}$, $a_{0,0}$, $-\tilde{a}$, $-\tilde{b}$, $\tilde{\phi}$ and x, respectively, hold, so that if $\tilde{a} - \tilde{b} \notin \mathbb{Z}$,

$$\tilde{\phi}_{\tilde{a}}(x) = x^{\tilde{a}} \cdot {}_0F_1(;1 + \tilde{a} - \tilde{b}; -\frac{a_{0,0}}{a_{2,3}}x), \quad \tilde{\phi}_{\tilde{b}}(x) = x^{\tilde{b}} \cdot {}_0F_1(;1 - \tilde{a} + \tilde{b}; -\frac{a_{0,0}}{a_{2,3}}x). \tag{3.20}$$

We put $a_{0,0} = 1$, $a_{2,3} = -\delta$, $z = t$, $\zeta = z^{-1}$, $\tilde{w}(\zeta) = v(x)$ and $w(z) = u(t)$. Then (2.2-0) and (3.3-1) are expressed by

$$w - \delta \cdot z[z^2 \cdot \frac{d^2w}{dz^2} + (1 + \tilde{a} + \tilde{b})z \cdot \frac{dw}{dz} + \tilde{a}\tilde{b}w] = 0, \tag{3.21}$$

$$-\delta[\zeta^2 \cdot \frac{d^2\tilde{w}}{d\zeta^2} + (1 - \tilde{a} - \tilde{b})\zeta \cdot \frac{d\tilde{w}}{d\zeta} + \tilde{a}\tilde{b}\tilde{w}] + \zeta\tilde{w} = 0, \tag{3.22}$$

and if $\tilde{a} - \tilde{b} \notin \mathbb{Z}$, we obtain the following solutions of (3.21), with the aid of (3.20):

$$\tilde{\Phi}_{\tilde{a}}(z) := \delta^{-\tilde{a}}\tilde{\phi}_{\tilde{a}}(x) = (\delta z)^{-\tilde{a}} \cdot {}_0F_1(;1 + \tilde{a} - \tilde{b}; (\delta z)^{-1}), \tag{3.23}$$

$$\tilde{\Phi}_{\tilde{b}}(z) := \delta^{-\tilde{b}}\tilde{\phi}_{\tilde{b}}(x) = (\delta z)^{-\tilde{b}} \cdot {}_0F_1(;1 - \tilde{a} + \tilde{b}; (\delta z)^{-1}). \tag{3.24}$$

Remark 3.5. We note that the solutions of (3.22) for the cases of (i): $n \in \mathbb{Z}_{>-1}$, $\tilde{a} = \tilde{b} - n + \epsilon$ and $\delta = 1$, (ii): $n \in \mathbb{Z}_{>-1}$, $\tilde{a} = \tilde{b} - n$ and $\delta = 1$, and (iii): $\tilde{a} = -1$, $\tilde{b} = 0$ and $\delta = 1$, are obtained from Remark 3.4 and Lemmas 3.3 and 3.4, respectively, with z, $w_1(z)$ and $w_2(z)$ replaced by ζ, $\tilde{w}_1(\zeta)$ and $\tilde{w}_2(\zeta)$, respectively. As a consequence, we have the following lemma from Lemma 3.4.

Lemma 3.5. *When $\tilde{a} = -1$, $\tilde{b} = 0$ and $\delta = 1$, Equation* (3.21) *is expressed by*

$$z^3 \cdot \frac{d^2w}{dz^2} - w = 0, \tag{3.25}$$

which has an irregular singular point at $z = 0$, and the solutions of (3.25) *are given by*

$$w_1(z) = \tilde{w}_1(\zeta) = {}_0F_1(;2;z^{-1}), \tag{3.26}$$

$$w_2(z) = \tilde{w}_2(\zeta) = z - {}_0F_1(;2;z^{-1})\log z - \sum_{k=1}^{\infty} \frac{\phi(k) + \phi(k+1) - 1}{k!(k+1)!} z^{-k}, \tag{3.27}$$

where $\phi(k)$ is given by (3.16)*.*

3.3 Asymptotic behaviors of the solutions of (2.1-2) for $|z| \to \infty$

We consider Equation (2.1-2) with $a_{0,1} \neq 0$ or $a_{0,0} \neq 0$ or both, for which we have the pair of solutions given in (2.7). In this case, Lemma 3.2 shows that (3.4-1) holds, and Remark 3.1 shows that Equations (2.2-1) and (2.12), with $a_{2,3}$, $a_{1,1}$, $a_{0,0}$, $\tilde{a}$, $\tilde{b}$, c, ψ and t replaced by $a_{2,2}$, $-a_{1,2}$, $a_{0,1}$, $-a$, $-b$, $-\tilde{c}$, $\tilde{\psi}$ and x, respectively, hold, so that

$$\tilde{\psi}_{\tilde{c}}(x) = x^{\tilde{c}} \cdot {}_2F_0(\tilde{c}-a, \tilde{c}-b; ; \frac{a_{2,2}}{a_{1,2}}x). \tag{3.28}$$

We put $a_{2,2} = 1$, $a_{1,2} = -\delta$, $z = t$, $\zeta = z^{-1}$, $w(z) := u(t)$ and $\tilde{w}(\zeta) := v(x)$. Then Equations (2.1-2) and (3.4-1) are expressed by

$$z^2 \cdot \frac{d^2 w}{dz^2} + (1+a+b)z \cdot \frac{dw}{dz} + abw - \delta z[z \cdot \frac{dw}{dz} + \tilde{c}w] = 0, \tag{3.29}$$

$$\delta[\zeta \cdot \frac{d\tilde{w}}{d\zeta} - \tilde{c}\tilde{w}] + \zeta^3 \cdot \frac{d^2\tilde{w}}{d\zeta^2} + (1-a-b)\zeta^2 \cdot \frac{d\tilde{w}}{d\zeta} + ab\zeta\tilde{w} = 0, \tag{3.30}$$

and if $a - b \notin \mathbb{Z}$, we obtain the following solutions of (3.29), with the aid of (2.7) and (3.28):

$$\Phi_{-a}(z) := \delta^{-a}\phi_{-a}(t) = (\delta z)^{-a} \cdot {}_1F_1(\tilde{c}-a; 1-a+b; \delta z), \tag{3.31}$$

$$\Phi_{-b}(z) := \delta^{-b}\phi_{-b}(t) = (\delta z)^{-b} \cdot {}_1F_1(\tilde{c}-b; 1-b+a; \delta z), \tag{3.32}$$

$$\tilde{\Psi}_{-\tilde{c}}(z) := \delta^{-\tilde{c}}\tilde{\psi}_{\tilde{c}}(x) = (\delta z)^{-\tilde{c}} \cdot {}_2F_0(\tilde{c}-a, \tilde{c}-b; ; -(\delta z)^{-1}). \tag{3.33}$$

When $b = 0$, $A = \tilde{c}$, $B = 1 + a$ and $\delta = 1$, (3.29) is Kummer's equation:

$$z \cdot \frac{d^2 w}{dz^2} + B \cdot \frac{dw}{dz} - z \cdot \frac{dw}{dz} - Aw = 0, \tag{3.34}$$

and its solution given by (3.32) becomes ${}_1F_1(A; B; z)$. In [5], the asymptotic behavior of ${}_1F_1(A; B; z)$ is given as follows.

Lemma 3.6. *Let $A \in \mathbb{C}\backslash\mathbb{Z}_{<1}$, $B \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ and $B - A \in \mathbb{C}\backslash\mathbb{Z}_{<1}$. Then*

$$ {}_1F_1(A; B; z) = e^z \cdot {}_1F_1(B-A; B; -z) \tag{3.35}$$

$$= e^{iA\pi}\frac{\Gamma(B)}{\Gamma(B-A)}U(A; B; z) + e^{i(B-A)\pi}\frac{\Gamma(B)}{\Gamma(A)}e^z U(B-A; B; e^{-i\pi}z), \tag{3.36}$$

where

$$U(A; B; z) := z^{-A} \cdot {}_2F_0(1+A-B, A; ; -z^{-1}), \tag{3.37}$$

$$e^{i(B-A)\pi}e^z U(B-A; B; e^{-i\pi}z) = e^z z^{A-B} \cdot {}_2F_0(1-A, B-A; ; z^{-1}). \tag{3.38}$$

Remark 3.6. In Section 13.5.2 of [8], (3.37) is stated to hold valid, when $-\frac{3}{2}\pi < \arg z < \frac{3}{2}\pi$. As a consequence, (3.38) holds valid, when $-\frac{1}{2}\pi < \arg z < \frac{5}{2}\pi$, and (3.36) with (3.37) and (3.38) hold valid, when $-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$. In Section 13.5.1 of [8], (3.36) is stated to hold valid when $-\frac{3}{2}\pi < \arg z < -\frac{1}{2}\pi$, if i in these equations are replaced by $-i$.

Remark 3.7. Equations (3.36) and (3.38) are taken from Equations (70) and (72) in [5], with corrections. The corrections are such that "i" in the first factor in the second term on the righthand side of (70) and in the first factor on the righthand side of (72) should be replaced by "$-i$".

With the aid of this lemma, we obtain the asymptotic behavior of the solutions (3.31) and (3.32) of Equation (3.29) as follows.

Lemma 3.7. *Let $\tilde{c}-a \in \mathbb{C}\backslash\mathbb{Z}_{<1}$, $1-a+b \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ and $1+b-\tilde{c} \in \mathbb{C}\backslash\mathbb{Z}_{<1}$. Then*

$$z^{-a} \cdot {}_1F_1(\tilde{c}-a; 1-a+b; z) = e^z z^{-a} \cdot {}_1F_1(1+b-\tilde{c}; 1-a+b; -z) \tag{3.39}$$

$$= e^{i(\tilde{c}-a)\pi} z^{-a} \frac{\Gamma(1-a+b)}{\Gamma(1+b-\tilde{c})} U(\tilde{c}-a; 1-a+b; z) + e^{i(1+b-\tilde{c})\pi} \frac{\Gamma(1-a+b)}{\Gamma(\tilde{c}-a)} e^z z^{-a} U(1+b-\tilde{c}; 1-a+b; e^{-i\pi} z), \tag{3.40}$$

where

$$z^{-a} U(\tilde{c}-a; 1-a+b; z) = z^{-\tilde{c}} \cdot {}_2F_0(\tilde{c}-b, \tilde{c}-a; ; -z^{-1}), \tag{3.41}$$

$$e^{i(1+b-\tilde{c})\pi} e^z z^{-a} U(1+b-\tilde{c}; 1-a+b; e^{-i\pi} z) = e^z z^{\tilde{c}-1-b-a} \cdot {}_2F_0(1-\tilde{c}+a, 1+b-\tilde{c}; ; z^{-1}). \tag{3.42}$$

Lemma 3.8. *Lemma 3.7 in which a and b are exchanged holds.*

Comparing (3.41) and (3.42) with the corresponding equations in Lemma 3.8, we see that the lefthand sides of these equations are equal with themselves with a and b exchanged. When we put $A = \tilde{c} - b$ and $B = 1 + a - b$, the equations showing these are expressed by

$$U(A; B; z) = z^{1-B} U(1+A-B; 2-B; z), \tag{3.43}$$

$$e^{i(B-A)\pi} e^z U(B-A; B; e^{-i\pi} z) = e^{i(1-A)\pi} e^z z^{1-B} U(1-A; 2-B; e^{-i\pi} z), \tag{3.44}$$

See Sections 13.1.29 and 13.1.30 in [8].

In discussing the asymptotic behaviors of the solutions given in Lemmas 3.6∼3.8, we use the following lemma; See Sections 13.5.3 and 13.5.4 of [8].

Lemma 3.9. *Let $0 < \epsilon \ll 1$, ${}_2F_0(a, b; ; z)$ be the asymptotic expansion of a function $f(z)$ near the origin, and $f_N(z) := \sum_{k=0}^{N} \frac{(a)_k (b)_k}{k!} z^k$ for $N \in \mathbb{Z}_{>0}$. Then there exist $N \in \mathbb{Z}_{>0}$ and $r \in \mathbb{R}_{>0}$, for which $|f_N(z) - f(z)| < \epsilon$, if $|z| < r$.*

Based on this lemma, we obtain

Remark 3.8. For the solutions given by (3.35) and (3.39) and the corrsesponding solution in Lemma 3.8, (3.36) and (3.40) show that Stokes' phenomenon occurs; See Section 7.22 in [9] for this phenomenon. For instance, the asymptotic behaviors given by (3.37) and (3.38), which are $U(A, B, z)$ and $e^z U(B-A, B, e^{-i\pi} z)$, are observed for the solution given by (3.35) as $|z| \to \infty$, for Re $z < 0$ and Re $z > 0$, respectively.

3.4 Solutions and asymptotic solutions of (2.2-1) for $|z| \to \infty$ and $|z| \to 0$, respectively

We consider Equation (2.2-1) with $a_{0,1} \neq 0$ or $a_{0,0} \neq 0$ or both, for which we have the solution given by (2.12). In this case, Lemma 3.2 shows that (3.3-2) holds, and Remark 3.1 shows that (2.1-2) and (2.7), with $a_{2,2}$, $a_{1,2}$, $a_{0,1}$, a, b, $\tilde{c}$, ϕ and t replaced by $a_{2,3}$, $-a_{1,1}$, $a_{0,0}$, $-\tilde{a}$, $-\tilde{b}$, $-c$, $\tilde{\phi}$ and x, respectively, hold, so that

$$\tilde{\phi}_{\tilde{a}}(x) = x^{\tilde{a}} \cdot {}_1F_1(\tilde{a}-c; 1+\tilde{a}-\tilde{b}; \frac{a_{1,1}}{a_{2,3}} x), \quad \tilde{\phi}_{\tilde{b}}(x) = x^{\tilde{b}} \cdot {}_1F_1(\tilde{b}-c; 1-\tilde{a}+\tilde{b}; \frac{a_{1,1}}{a_{2,3}} x). \tag{3.45}$$

We put $a_{2,3} = \delta$, $a_{1,1} = 1$, $z = t$, $\zeta = z^{-1}$, $\tilde{w}(\zeta) = v(x)$ and $w(z) = u(t)$. Then (2.2-1) and (3.3-2) are expressed by

$$z \cdot \frac{dw}{dz} + cw + \delta z[z^2 \cdot \frac{d^2w}{dz^2} + (1+\tilde{a}+\tilde{b})z \cdot \frac{dw}{dz} + \tilde{a}\tilde{b}w] = 0, \tag{3.46}$$

$$\delta[\zeta^2 \cdot \frac{d^2\tilde{w}}{d\zeta^2} + (1-\tilde{a}-\tilde{b})\zeta \cdot \frac{d\tilde{w}}{d\zeta} + \tilde{a}\tilde{b}\tilde{w}] - \zeta^2 \cdot \frac{d\tilde{w}}{d\zeta} + c\zeta\tilde{w} = 0, \tag{3.47}$$

and we obtain the following solutions of (3.46), with the aid of (2.12) and (3.45):

$$\Psi_{-c}(z) := \delta^{-c}\psi_{-c}(t) = (\delta z)^{-c} \cdot {}_2F_0(\tilde{a}-c, \tilde{b}-c;;-\delta z), \tag{3.48}$$

$$\tilde{\Phi}_{-\tilde{a}}(z) := \delta^{-\tilde{a}}\tilde{\phi}_{\tilde{a}}(x) = (\delta z)^{-\tilde{a}} \cdot {}_1F_1(\tilde{a}-c; 1+\tilde{a}-\tilde{b}; (\delta z)^{-1}), \tag{3.49}$$

$$\tilde{\Phi}_{-\tilde{b}}(z) := \delta^{-\tilde{b}}\tilde{\phi}_{\tilde{b}}(x) = (\delta z)^{-\tilde{b}} \cdot {}_1F_1(\tilde{b}-c; 1-\tilde{a}+\tilde{b}; (\delta z)^{-1}). \tag{3.50}$$

When $\tilde{b} = 0$, $\tilde{A} = -c$, $\tilde{B} = 1-\tilde{a}$ and $\delta = 1$, (3.47) is Kummer's equation:

$$\zeta \cdot \frac{d^2\tilde{w}}{d\zeta^2} + \tilde{B} \cdot \frac{d\tilde{w}}{d\zeta} - \zeta \cdot \frac{d\tilde{w}}{d\zeta} - \tilde{A}\tilde{w} = 0, \tag{3.51}$$

and the solution given by (3.50) becomes ${}_1F_1(\tilde{A}; \tilde{B}; z^{-1})$. The asymptotic behavior near the origin of ${}_1F_1(\tilde{A}; \tilde{B}; z^{-1})$ is obtained by replacing A, B and z in Lemma 3.6 by $\tilde{A}$, $\tilde{B}$ and z^{-1}, respectively, as follows.

Lemma 3.10. *Let* $\tilde{A} \in \mathbb{C}\backslash\mathbb{Z}_{<1}$, $\tilde{B} \in \mathbb{C}\backslash\mathbb{Z}_{<1}$ *and* $\tilde{B} - \tilde{A} \in \mathbb{C}\backslash\mathbb{Z}_{<1}$. *Then*

$${}_1F_1(\tilde{A}; \tilde{B}; z^{-1}) = e^{1/z}{}_1F_1(\tilde{B}-\tilde{A}; \tilde{B}; -z^{-1}) \tag{3.52}$$

$$= e^{i\tilde{A}\pi}\frac{\Gamma(\tilde{B})}{\Gamma(\tilde{B}-\tilde{A})}U(\tilde{A}; \tilde{B}; z^{-1}) + e^{i(\tilde{B}-\tilde{A})\pi}\frac{\Gamma(\tilde{B})}{\Gamma(\tilde{A})}e^{1/z}U(\tilde{B}-\tilde{A}; \tilde{B}; e^{-i\pi}z^{-1}), \tag{3.53}$$

where

$$U(\tilde{A}; \tilde{B}; z^{-1}) = z^{\tilde{A}} \cdot {}_2F_0(1+\tilde{A}-\tilde{B}, \tilde{A};;-z), \tag{3.54}$$

$$e^{i(\tilde{B}-\tilde{A})\pi}e^{1/z}U(\tilde{B}-\tilde{A}; \tilde{B}; e^{-i\pi}z^{-1}) = e^{1/z}z^{-\tilde{A}+\tilde{B}} \cdot {}_2F_0(1-\tilde{A}, \tilde{B}-\tilde{A};;z). \tag{3.55}$$

Remark 3.9. In Remark 3.6, it was stated that (3.36) with (3.37) and (3.38) holds valid for $-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$, and hence (3.53) with (3.54) and (3.55) holds for for $-\frac{1}{2}\pi < \arg(z^{-1}) < \frac{3}{2}\pi$ so that for $-\frac{3}{2}\pi < \arg z < \frac{1}{2}\pi$. If all i in (3.53)∼(3.55) are replaced by $-i$, (3.53) with (3.54) and (3.55) are valid for $-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$.

We note that we obtain the asymptotic behavior near the origin of the solutions given by (3.49), by replacing a, b, $\tilde{c}$ and z in Lemma 3.7 by $-\tilde{a}$, $-\tilde{b}$, $-c$ and z^{-1}, respectively, as follows.

Lemma 3.11. *Let* $-c+\tilde{a} \in \mathbb{C}\backslash\mathbb{Z}_{<1}$, $1+\tilde{a}-\tilde{b} \in \mathbb{C}\backslash\mathbb{Z}_{<1}$, *and* $1-\tilde{b}+c \in \mathbb{C}\backslash\mathbb{Z}_{<1}$. *Then*

$$z^{-\tilde{a}} \cdot {}_1F_1(-c+\tilde{a}; 1+\tilde{a}-\tilde{b}; z^{-1}) = e^{1/z}z^{-\tilde{a}} \cdot {}_1F_1(1-\tilde{b}+c; 1+\tilde{a}-\tilde{b}; -z^{-1}) \tag{3.56}$$

$$= e^{i(-c+\tilde{a})\pi}z^{-\tilde{a}}\frac{\Gamma(1+\tilde{a}-\tilde{b})}{\Gamma(1-\tilde{b}+c)}U(-c+\tilde{a}; 1+\tilde{a}-\tilde{b}; z^{-1})$$

$$+e^{i(1-\tilde{b}+c)\pi}\frac{\Gamma(1+\tilde{a}-\tilde{b})}{\Gamma(-c+\tilde{a})}e^{1/z}z^{-\tilde{a}}U(1-\tilde{b}+c; 1+\tilde{a}-\tilde{b}; e^{-i\pi}z^{-1}), \tag{3.57}$$

where

$$z^{-\tilde{a}}U(-c+\tilde{a}; 1+\tilde{a}-\tilde{b}; z^{-1}) = z^{-c} \cdot {}_2F_0(-c+\tilde{b}, -c+\tilde{a};;-z), \tag{3.58}$$

$$e^{i(1-\tilde{b}+c)\pi}e^{1/z}z^{-\tilde{a}}U(1-\tilde{b}+c; 1+\tilde{a}-\tilde{b}; e^{-i\pi}z^{-1}) = e^{1/z}z^{c+1-\tilde{b}-\tilde{a}} \cdot {}_2F_0(1+c-\tilde{a}, 1-\tilde{b}+c;;z). \tag{3.59}$$

Lemma 3.12. *Lemma 3.11 in which $\tilde{a}$ and $\tilde{b}$ are exchanged holds.*

Remark 3.10. We note that the three equations (3.48)∼(3.50) correspond to (3.31)∼(3.33) in Section 3.3. The behaviors at $|z| \to \infty$ and $|z| \to 0$, of the second set of equations, must describe the behaviors at $|z| \to 0$ and $|z| \to \infty$, respectively, of the first set of equations. In particular, from the Stokes' phenomenon described in Remark 3.8 for the second set, we expect Stokes' phenomenon at $|z| \to 0$ for the first set as follows. Asymptotic behaviors for $U(\tilde{A}, \tilde{B}, z^{-1})$ and $e^{1/z}U(\tilde{B} - \tilde{A}, \tilde{B}, e^{-i\pi}z^{-1})$ are observed for the solution given by (3.52) as $|z| \to 0$ for Re $z < 0$ and Re $z > 0$, respectively.

3.5 Behaviors of the solutions of (2.1-3) for $|z| \to \infty$

We consider Equation (2.1-3) with $a_{0,1} \neq 0$ or $a_{0,0} \neq 0$ or both, for which we have the pair of solutions in (2.6). In this case, Lemma 3.2 shows that Equation (3.4-2) holds, and Remark 3.1 shows that (2.1-3) and (2.6), with $a_{2,3}$, $a_{2,2}$, a, b, $\tilde{a}$, $\tilde{b}$, ϕ and t replaced by $a_{2,2}$, $a_{2,3}$, $-\tilde{a}$, $-\tilde{b}$, $-a$, $-b$, $\tilde{\phi}$ and x, respectively, hold, so that if $\tilde{a} - \tilde{b} \notin \mathbb{Z}$,

$$\tilde{\phi}_{\tilde{a}}(x) = x^{\tilde{a}} \cdot {}_2F_1(\tilde{a} - a, \tilde{a} - b; 1 + \tilde{a} - \tilde{b}; -\frac{a_{2,2}}{a_{2,3}}x),$$
$$\tilde{\phi}_{\tilde{b}}(x) = x^{\tilde{b}} \cdot {}_2F_1(\tilde{b} - a, \tilde{b} - b; 1 - \tilde{a} + \tilde{b}; -\frac{a_{2,2}}{a_{2,3}}x). \quad (3.60)$$

We put $a_{2,2} = 1$, $a_{2,3} = -\delta$, $z = t$, $\zeta = z^{-1}$, $w(z) = u(t)$ and $\tilde{w}(\zeta) = v(x)$. Then Equations (2.1-3) and (3.4-2) are expressed by

$$z^2 \cdot \frac{d^2w}{dz^2} + (1 + a + b)z \cdot \frac{dw}{dz} + abw - \delta z[z^2 \cdot \frac{d^2w}{dz^2} + (1 + \tilde{a} + \tilde{b})z \cdot \frac{dw}{dz} + \tilde{a}\tilde{b}w] = 0, \quad (3.61)$$
$$-\delta[\zeta^2 \cdot \frac{d^2\tilde{w}}{d\zeta^2} + (1 - \tilde{a} - \tilde{b})\zeta \cdot \frac{d\tilde{w}}{d\zeta} + \tilde{a}\tilde{b}\tilde{w}] + \zeta^3 \cdot \frac{d^2\tilde{w}}{d\zeta^2} + (1 - a - b)\zeta^2 \cdot \frac{d\tilde{w}}{d\zeta} + ab\zeta\tilde{w} = 0, \quad (3.62)$$

and if $a - b \notin \mathbb{Z}$ and $\tilde{a} - \tilde{b} \notin \mathbb{Z}$, we obtain the following solutions of (3.61), with the aid of (2.6) and (3.60):

$$\Phi_{-a}(z) = \delta^{-a} \cdot \phi_{-a}(t) = (\delta z)^{-a} \cdot {}_2F_1(\tilde{a} - a, \tilde{b} - a; 1 - a + b; \delta z), \quad (3.63)$$
$$\Phi_{-b}(z) = \delta^{-b} \cdot \phi_{-b}(t) = (\delta z)^{-b} \cdot {}_2F_1(\tilde{a} - b, \tilde{b} - b; 1 + a - b; \delta z), \quad (3.64)$$

$$\tilde{\Phi}_{-\tilde{a}}(z) = \delta^{-\tilde{a}} \cdot \tilde{\phi}_{\tilde{a}}(x) = (\delta z)^{-\tilde{a}} \cdot {}_2F_1(\tilde{a} - a, \tilde{a} - b; 1 + \tilde{a} - \tilde{b}; (\delta z)^{-1}), \quad (3.65)$$
$$\tilde{\Phi}_{-\tilde{b}}(z) = \delta^{-\tilde{b}} \cdot \tilde{\phi}_{\tilde{b}}(x) = (\delta z)^{-\tilde{b}} \cdot {}_2F_1(\tilde{b} - a, \tilde{b} - b; 1 - \tilde{a} + \tilde{b}; (\delta z)^{-1}). \quad (3.66)$$

When $b = 0$, $A = \tilde{a}$, $B = \tilde{b}$, $C = 1 + a$ and $\delta = 1$, (3.61) is the hypergeometric differential equation:

$$z(1 - z)\frac{d^2w}{dz^2} + [C - (1 + A + B)z]\frac{dw}{dz} - ABw = 0. \quad (3.67)$$

and if $C \notin \mathbb{Z}_{<0}$, its solution (3.64) becomes ${}_2F_1(A, B; C; z)$,

In Section 15.3.7 of [8], and in Section 2.4.1 of [7], we have the following formula, which gives the behavior near infinity of ${}_2F_1(A, B; C; z)$:

$${}_2F_1(A, B; C; z) = \frac{\Gamma(C)\Gamma(B - A)}{\Gamma(B)\Gamma(C - A)}(-z)^{-A} \cdot {}_2F_1(A + 1 - C, A; 1 + A - B; z^{-1})$$
$$+ \frac{\Gamma(C)\Gamma(A - B)}{\Gamma(A)\Gamma(C - B)}(-z)^{-B} \cdot {}_2F_1(B + 1 - C, B; 1 - A + B; z^{-1}), \quad (3.68)$$

for $|z| > 1$ and $\arg(-z) < \pi$.

Remark 3.11. This formula can be derived by the method used in deriving Lemma 3.6 in [5]. The derivation is given in next section.

With the aid of this lemma, we obtain the behavior near infinity of the solutions (3.63) and (3.64) of Equation (3.61) as follows.

Lemma 3.13. *Let* $\tilde{c} - a \in \mathbb{C}\backslash\mathbb{Z}_{<1}$, $1 - a + b \in \mathbb{C}\backslash\mathbb{Z}_{<1}$, *and* $1 + b - \tilde{c} \in \mathbb{C}\backslash\mathbb{Z}_{<1}$. *Then*

$$z^{-a} \cdot {}_2F_1(\tilde{a} - a, \tilde{b} - a; 1 - a + b; z) = \frac{\Gamma(1 - a + b)\Gamma(\tilde{b} - \tilde{a})}{\Gamma(\tilde{b} - a)\Gamma(1 + b - \tilde{a})}(-z)^{-\tilde{a}} \cdot {}_2F_1(\tilde{a} - b, \tilde{a} - a; 1 + \tilde{a} - \tilde{b}; z^{-1}) + \frac{\Gamma(1 - a + b)\Gamma(\tilde{a} - \tilde{b})}{\Gamma(\tilde{a} - a)\Gamma(1 + b - \tilde{b})}(-z)^{-\tilde{b}} \cdot {}_2F_1(\tilde{b} - b, \tilde{b} - a; 1 - \tilde{a} + \tilde{b}; z^{-1}), \tag{3.69}$$

for $|z| > 1$ *and* $\arg(-z) < \pi$.

Lemma 3.14. *Lemma 3.13 in which* a *and* b *are exchanged holds.*

4 Solutions of the Hypergeometric Equation with the Aid of Fractional Calculus

Lemma 4.1. *Let* $\xi = 0$ *or* $\xi = \infty$. *Then we have solutions of* (3.67) *given by*

$$w_1(\xi, z) = \gamma_1(\xi) \cdot {}_PD_\xi^{A-1}[z^{A-C}(1 - z)^{C-B-1}], \tag{4.1}$$

$$w_3(\xi, z) = \gamma_3(\xi) \cdot z^{1-C}\, {}_PD_\xi^{B-C}[z^{B-1}(1 - z)^{-A}], \tag{4.2}$$

in [10], where

$$\gamma_1(0) = \frac{\Gamma(2 - C)}{\Gamma(1 + A - C)}, \quad \gamma_3(0) = \frac{\Gamma(C)}{\Gamma(B)}, \tag{4.3}$$

$$\gamma_1(\infty) = \frac{\Gamma(1 - A + B)}{\Gamma(B)}, \quad \gamma_3(\infty) = \frac{\Gamma(1 - B + A)}{\Gamma(1 - C + A)}. \tag{4.4}$$

Here the fractional derivative ${}_PD_\xi^\nu[f(z)]$ *for* $\nu = -\lambda$ *satisfying* $\mathrm{Re}\,\lambda > 0$ *is equal to*

$$ {}_RD^{-\lambda}[f(z)] = \frac{1}{\Gamma(\lambda)} \int_\xi^z (z - \zeta)^{\lambda-1} f(\zeta) d\zeta, \tag{4.5}$$

when the integral exists, and ${}_PD_\xi^\nu[z^\kappa(1 - z)^\lambda]$ *is assumed to be analytic as a function of* ν, *of* κ *as well as of* λ; *See [11].*

Remark 4.1. In Table 1 in [10], a_l and b_l for $l = 3$ are $1 - c + a$ and $1 - c + b$, which should be replaced by $1 - c + b$ and $1 - c + a$, respectively. In Lemma 4.1, this correction is included.

In [11], ${}_PD_0^\nu f(z)$ is defined for a function $f(z)$ which is expressed by $f(z) = z^{\lambda-1} f_1(z)$, as follows.

Definition 4.1. Let $\nu \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash\mathbb{Z}$, $C_P(z)$ be Pohhammer's path of integration, that is $P_0 := (2\epsilon, -\epsilon) \to P_1 := z \cdot (1 - \epsilon, -\epsilon)$, z^+, $z \cdot (1 - \epsilon, 2\epsilon) \to (\epsilon, 2\epsilon)$, 0^+, $(\epsilon, \epsilon) \to z \cdot (1 - 2\epsilon, \epsilon)$, z^-, $z \cdot (1 - 2\epsilon, -2\epsilon) \to (2\epsilon, -\epsilon)$, 0^-, $(2\epsilon, -\epsilon)$, where $0 < \epsilon < \frac{1}{4}$, $\to$ denotes the path along the line segment from the preceding point to the succeeding point, and z^+ and z^- denote the pathes from the preceding point to the succeeding point, along a circle around the point z counter-clockwise and clockwise, respectively, and $f_1(z)$ be assumed to be analytic within the path $C_P(z)$. Then if $\nu \notin \mathbb{Z}_{<0}$,

$$ {}_PD_0^{\nu}[z^{\lambda-1}f_1(z)] = \frac{e^{-i\pi\lambda}\Gamma(\nu+1)}{4\pi\sin(\lambda\pi)}\int_{C_P(z)}(\zeta-z)^{-\nu-1}\zeta^{\lambda-1}f_1(\zeta)d\zeta $$
$$ = \frac{1}{\Gamma(-\nu)}\frac{-e^{i\pi\nu}e^{-i\pi\lambda}}{4\sin(-\nu\pi)\sin(\lambda\pi)}\int_{C_P(z)}(z-\zeta)^{-\nu-1}\zeta^{\lambda-1}f_1(\zeta)d\zeta, \tag{4.6} $$

and if $\nu = -n \in \mathbb{Z}_{<0}$ and $\kappa \in \mathbb{C}$, ${}_PD_0^{-n}[z^{\lambda-1}f_1(z)] = \lim_{\kappa\to n}{}_PD_0^{-\kappa}[z^{\lambda-1}f_1(z)]$.

Remark 4.2. Path $C_P(z)$ may be denoted by $C(P, z^+, 0^+, z^-, 0^-)$, where P is a point on the path from P_0 to P_1.

In the following proofs, we use

Lemma 4.2. *Let $\kappa \in \mathbb{C}$, $\lambda \in \mathbb{C}$, $A \in \mathbb{C}\backslash\mathbb{Z}_{<1}$, $k \in \mathbb{Z}_{>-1}$, $A' \in \mathbb{C}\backslash\mathbb{Z}$ and $B \in \mathbb{C}\backslash\mathbb{Z}_{<1}$. Then*

$$ B(\kappa,\lambda) = \int_0^1 (1-t)^{\kappa-1}t^{\lambda-1}dt = \frac{\Gamma(\kappa)\Gamma(\lambda)}{\Gamma(\kappa+\lambda)}, \quad \text{Re}\,\kappa > 0,\ \text{Re}\,\lambda > 0, \tag{4.7} $$
$$ B(\kappa,\lambda) = \frac{-e^{-i\pi\lambda}e^{-i\pi\kappa}}{4\sin(\lambda\pi)\sin(\kappa\pi)}\int_{C_P(1)}(1-\zeta)^{\kappa-1}\zeta^{\lambda-1}d\zeta = \frac{\Gamma(\kappa)\Gamma(\lambda)}{\Gamma(\kappa+\lambda)}, \quad \kappa\notin\mathbb{Z},\ \lambda\notin\mathbb{Z}. \tag{4.8} $$

$$ \Gamma(A+k) = \Gamma(A)(A)_k, \quad \Gamma(A'-k) = \frac{\Gamma(A')}{(-1)^k(-A'+1)_k}, \quad \binom{-B}{k} = \frac{(B)_k(-1)^k}{k!}, \tag{4.9} $$

See Section 12.43 of [12] for Equation (4.8).

Lemma 4.3. *Let $\Phi_{1-C}(z)$, $\Phi_0(z)$, $\tilde{\Phi}_{-A}(z)$ and $\tilde{\Phi}_{-B}(z)$ be given by* (3.63)∼(3.66) *for $b = 0$, $A = \tilde{a}$, $B = \tilde{b}$, $C = 1 + a$ and $\delta = 1$. Then*

$$ w_1(0,z) = \Phi_{1-C}(z), \quad w_1(\infty\cdot z, z) = (-1)^{C-A}\tilde{\Phi}_{-B}(z), \tag{4.10} $$
$$ w_3(0,z) = \Phi_0(z), \quad w_3(\infty\cdot z, z) = (-1)^{C-B}\tilde{\Phi}_{-A}(z). \tag{4.11} $$

Proof. Proofs are given for the two equations in (4.11). By (4.2) and (4.5),

$$ w_3(\xi,z) = \gamma_3(\xi)z^{1-C}\frac{1}{\Gamma(C-B)}\int_\xi^z (z-\zeta)^{C-B-1}\zeta^{B-1}(1-\zeta)^{-A}d\zeta, \tag{4.12} $$

if the integral on the righthand side exists. We put $\xi = 0$ and $\zeta = zx$. Then if Re $(C-B) > 0$, Re $B > 0$ and $z \neq 1$, we have

$$ w_3(0,z) = \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)}\int_0^1 (1-x)^{C-B-1}x^{B-1}(1-zx)^{-A}dx. \tag{4.13} $$

When $|z| < 1$, we have

$$ w_3(0,z) = \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)}\sum_{k=0}^{\infty}\binom{-A}{k}(-1)^k z^k\int_0^1(1-x)^{C-B-1}x^{B-1+k}dx $$
$$ = \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)}\sum_{k=0}^{\infty}\frac{(A)_k\Gamma(C-B)\Gamma(B+k)}{k!\Gamma(C+k)}z^k = \sum_{k=0}^{\infty}\frac{(A)_k(B)_k}{k!(C)_k}z^k = {}_2F_1(A,B;C;z). \tag{4.14} $$

When $\theta = \arg z$, $\xi = \infty \cdot e^{i\theta}$ and $|z| > 1$, we put $\zeta = \frac{1}{x} e^{i\theta}$. Then

$$\begin{aligned} w_3(\infty \cdot e^{i\theta}, z) &= \frac{\Gamma(1-B+A)}{\Gamma(1-C+A)\Gamma(C-B)} (-1)^{C-B} (-z)^{-A} \int_0^1 (1-x)^{C-B-1} x^{-C+A} (1-xz^{-1})^{-A} dx \\ &= \frac{\Gamma(1-B+A)}{\Gamma(1-C+A)\Gamma(C-B)} (-1)^{C-B} (-z)^{-A} \sum_{k=0}^{\infty} \frac{(A)_k \Gamma(C-B)\Gamma(1-C+A+k)}{k!\Gamma(1-B+A+k)} z^{-k} \\ &= (-1)^{C-B} (-z)^{-A} \cdot {}_2F_1(A+1-C, A; 1-B+A; z^{-1}). \end{aligned} \tag{4.15}$$

□

Lemma 4.4. *Let* Re $(C-B) > 0$, Re $B > 0$, $|z| > 1$ *and* $\arg(-z) < \pi$. *Then* $w_3(0, z)$ *given by* (4.13) *is expressed by*

$$w_3(0, z) = \frac{\Gamma(C)\Gamma(B-A)}{\Gamma(B)\Gamma(C-A)} (-z)^{-A} \cdot {}_2F_1(A-C+1, A; A-B+1; z^{-1}). \tag{4.16}$$

Proof. By using (4.13), we have

$$\begin{aligned} w_3(0; z) &= \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)} \int_0^1 (1-x)^{C-B-1} x^{B-1} (1-zx)^{-A} dx \\ &= \frac{\Gamma(C)(-z)^{-A}}{\Gamma(B)\Gamma(C-B)} \frac{-e^{-i\pi(C-B)} e^{-i\pi(B-A)}}{4\sin[\pi(C-B)]\sin[\pi(B-A)]} \int_{C_P(1)} (1-\zeta)^{C-B-1} \zeta^{B-1-A} (1 - \frac{1}{z\zeta})^{-A} d\zeta \\ &= \frac{\Gamma(C)(-z)^{-A}}{\Gamma(B)\Gamma(C-B)} \\ &\times \frac{-e^{-i\pi(C-B)} e^{-i\pi(B-A)}}{4\sin[\pi(C-B)]\sin[\pi(B-A)]} \sum_{k=0}^{\infty} \binom{-A}{k} (-1)^k z^{-k} \int_{C_P(1)} (1-\zeta)^{C-B-1} \zeta^{B-1-A-k} d\zeta \\ &= \frac{\Gamma(C)}{\Gamma(B)\Gamma(C-B)} (-z)^{-A} \sum_{k=0}^{\infty} \frac{(A)_k \Gamma(C-B)\Gamma(B-A-k)}{k!\Gamma(C-A-k)} z^{-k} \\ &= \frac{\Gamma(C)\Gamma(B-A)}{\Gamma(B)\Gamma(C-A)} (-z)^{-A} \sum_{k=0}^{\infty} \frac{(A)_k (A-C+1)_k}{k!(A-B+1)_k} z^{-k}. \end{aligned} \tag{4.17}$$

□

Remark 4.3. By the derivation of $w_3(0, z)$ given by (4.16) in Lemma 4.4, we see that it is an analytic continuation of ${}_2F_1(A, B; C; z)$ to the domain given by $|z| > 1$ and $\arg(-z) < \pi$. We note that ${}_2F_1(A, B; C; z)$ does not change when A and B are exchanged, and hence $w_3(0, z)$ obtained from (4.16) by the exchange of A and B, must be another analytic continuation of ${}_2F_1(A, B; C; z)$. The sum of these analytic continuations is expressed by (3.68), which is seen to be a linear combination of (3.65) and (3.66) for $\delta = 1$.

5 Conclusion

In [1], it is proposed to use the expression of the differential equation with polynomial coefficients, in terms of blocks of classified terms. The differential equation with only one block is called Euler's equation, which is easily solved. In [1], the solutions of the differential equation of two blocks are studied, where the solutions of the form (1.12) are expressed by the generalized hypergeometric functions. In particular, the solutions of Equation (1.10) of the second order are expressed in terms of the four functions given in (1.13) and (1.14). That part of the results is summarized in a theorem. In Section 2 of the present paper, we reproduce it as Theorem 2.1. In Section 3, we show that the solutions applicable near infinity are obtained with the aid of the solutions given in the theorem, and then discussion is given of the behaviors near infinity of the solutions given in the theorem.

In [5], the asymptotic behaviors as $t \to \infty$ are discussed for the confluent hypergeometric function, which is a solution of Kummer's differential equation, in the standpoint of fractional calculus, where the solutions are near the origin, are near infinity and asymptotic solutions are of the solutions near the origin as $t \to \infty$, for Kummer's equation. In Sections 3.3 and 3.4, we show that the solution near infinity that is of the form (1.15) is easily obtained from the solutions given in the theorem in Section 2, for Equation (1.10) for $l_x = 2$. Discussion is also given on the behaviors of the solution given in the theorem as $t \to \infty$. The results for the solutions of Equation (2.1-2) are obtained with the aid of the results in [5].

In [10], Kummer's 24 solutions of the hypergeometric equation are derived in the standpoint of fractional calculus. The results are used to discuss the behaviors of the solutions of Equation (2.1-3). In Section 4, by using the solution obtained by fractional calculus, an argument is given to show how the behavior near infinity is obtained from the solution near the origin.

In the present paper, we focussed attention to the solution of Equation (1.10). In the preceding paper, it was mentioned that the solutions of an equation described by (1.9) for $m > 1$ are obtained from those of an equation described by (1.10) by a change of variable.

In Section 3.2, we derive a solution which is not expressed in the form (1.12) for Equation (1.16), as an example.

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Note

After publishing [1], Author KS asked Author TM, how Equation (3.25) is solved. Author TM wrote a preliminary manuscript of this paper to answer this question.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Morita T, Sato K. A study on the solution of linear differential equations with polynomial coefficients. J. Adv. Math. Comput. Sci. 2018;28(3):1-15.

[2] Morita T, Sato K. Solution of Laplace's differential equation and fractional differential equation of that type. Applied Math. 2013;4(11A):26-36.

[3] Morita T, Sato K. Solution of differential equations with the aid of an analytic continuation of Laplace transform. Applied Math. 2014;5:1209-1219.

[4] Morita T, Sato K. Solution of differential equations with polynomial coefficients with the aid of an analytic continuation of Laplace transform. Mathematics. 2016;4(19):1-18.

[5] Morita T, Sato K. Asymptotic expansions of fractional derivatives and their applications. Mathematics. 2015;3:171-189.

[6] Ince EL. Ordinary differential equations. Dover Publ. Inc., New York; 1956.

[7] Magnus M, Oberhettinger F, Soni RP. Formulas and theorems for the functions of mathematical physics. Springer-Verlag New York Inc., New York; 1966.

[8] Abramowitz M, Stegun IA. Handbook of mathematical functions with formulas, graphs and mathematical tables. Dover Publ. Inc., New York; 1972.

[9] Watson GN. A treatise on the theory of Bessel functions. Cambridge U.P., Cambridge; 1922.

[10] Morita T, Sato K. Kummer's 24 solutions of the hypergeometric equation with the aid of fractional calculus. Adv. Pure Math. 2016;6:180-191.

[11] Morita T, Sato K. Liouville and Riemann-Liouville fractional derivatives via contour integrals. Frac. Calc. Appl. Anal. 2013;16:630-653.

[12] Whittaker ET, Watson GN. A course of modern analysis. Cambridge U.P., Cambridge; 1935.

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