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On Generalized Third-Order Pell Numbers

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, we investigate the generalized third order Pell sequences and we deal with, in detail, three special cases which we call them third order Pell, third order Pell-Lucas and modified third order Pell sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

Keywords: Pell numbers; third order Pell numbers; third order Pell-Lucas numbers.

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1 INTRODUCTION

In this paper, we introduce the generalized third order Pell sequences and we investigate, in detail, three special case which we call them third order Pell, third order Pell-Lucas and modified third order Pell sequences.

It is well-known that the Pell sequence (sequence A000129 in [1]) $\{P_n\}$ is defined recursively by the equation, for $n \geq 0$

$$P_{n+2} = 2P_{n+1} + P_n$$

in which $P_0 = 0$ and $P_1 = 1$. Then Pell sequence (second order Pell sequence) is

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$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, ...$$

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [2,3,4,5,6,7,8,9,10]. For higher order Pell sequences, see [11,12].

The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \ \ n \geq 3 \tag{1.1}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers.

This sequence has been studied by many authors, see for example [13-25].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer $n.$

As $\{W_n\}$ is a third order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{1.2}$$

whose roots are

$$\begin{aligned} \alpha &= \alpha(r, s, t) = \frac{r}{3} + A + B \\ \beta &= \beta(r, s, t) = \frac{r}{3} + \omega A + \omega^2 B \\ \gamma &= \gamma(r, s, t) = \frac{r}{3} + \omega^2 A + \omega B \end{aligned}$$

where

$$\begin{aligned} A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \ B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3} \\ \Delta &= \Delta(r, s, t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \ \ \omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3) \end{aligned}$$

Note that we have the following identities

$$\begin{aligned} \alpha + \beta + \gamma &= r, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -s, \\ \alpha\beta\gamma &= t. \end{aligned}$$

If $\Delta(r, s, t) > 0$, then the Equ. (1.2) has one real (α) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that generalized Tribonacci numbers can be expressed, for all integers n, using Binet's formula

$$W_n = \frac{b_1\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{b_2\beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{b_3\gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \tag{1.3}$$

where

$$b_1 = W_2 - (\beta + \gamma)W_1 + \beta\gamma W_0, \ b_2 = W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0, \ b_3 = W_2 - (\alpha + \beta)W_1 + \alpha\beta W_0.$$

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n, for a proof of this result see [26]. This result of Howard and Saidak [26] is even true in the case of higher-order recurrence relations.

In this paper we consider the case $r = 2$, $s = t = 1$ and in this case we write $V_n = W_n$. A generalized third order Pell sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3} \tag{1.4}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - 2V_{-(n-2)} + V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.4) holds for all integer n.

(1.3) can be used to obtain Binet formula of generalized third order Pell numbers. Binet formula of generalized third order Pell numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \tag{1.5}$$

Here, α, β and $\gamma$are the roots of the cubic equation $x^3 - 2x^2 - x - 1 = 0$. Moreover

$$\begin{aligned}
\alpha &= \frac{2}{3} + \left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3} \\
\beta &= \frac{2}{3} + \omega\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega^2\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3} \\
\gamma &= \frac{2}{3} + \omega^2\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3)$$

Note that

$$\begin{aligned}
\alpha + \beta + \gamma &= 2, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

The first few generalized third order Pell numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized third order Pell numbers

n	V_n	V_{-n}
0	V_0	V_0
1	V_1	$-V_0 - 2V_1 + V_2$
2	V_2	$-V_2 + 3V_1 - V_0$
3	$2V_2 + V_1 + V_0$	$-V_2 + V_1 + 4V_0$
4	$5V_2 + 3V_1 + 2V_0$	$4V_2 - 9V_1 - 3V_0$
5	$13V_2 + 7V_1 + 5V_0$	$-3V_2 + 10V_1 - 6V_0$
6	$33V_2 + 18V_1 + 13V_0$	$-6V_2 + 9V_1 + 16V_0$
7	$84V_2 + 46V_1 + 33V_0$	$16V_2 - 38V_1 - 7V_0$
8	$117V_1 + 214V_2 + 84V_0$	$-7V_2 + 30V_1 - 31V_0$

Now we define three special case of the sequence $\{V_n\}$. Third-order Pell sequence$\{P_n^{(3)}\}_{n\geq 0}$, third-order Pell-Lucas sequence $\{Q_n^{(3)}\}_{n\geq 0}$ and modified third-order Pell sequence $\{E_n^{(3)}\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$P_{n+3}^{(3)} = 2P_{n+2}^{(3)} + P_{n+1}^{(3)} + P_n^{(3)}, \quad P_0^{(3)} = 0, P_1^{(3)} = 1, P_2^{(3)} = 2, \tag{1.6}$$

$$Q_{n+3}^{(3)} = 2Q_{n+2}^{(3)} + Q_{n+1}^{(3)} + Q_n^{(3)}, \quad Q_0^{(3)} = 3, Q_1^{(3)} = 2, Q_2^{(3)} = 6 \tag{1.7}$$

and

$$E_{n+3}^{(3)} = 2E_{n+2}^{(3)} + E_{n+1}^{(3)} + E_n^{(3)}, \quad E_0^{(3)} = 0, E_1^{(3)} = 1, E_2^{(3)} = 1. \tag{1.8}$$

The sequences $\{P_n^{(3)}\}_{n\geq 0}$, $\{Q_n^{(3)}\}_{n\geq 0}$ and $\{E_n^{(3)}\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n}^{(3)} = -P_{-(n-1)}^{(3)} - 2P_{-(n-2)}^{(3)} + P_{-(n-3)}^{(3)} \tag{1.9}$$

and

$$Q_{-n}^{(3)} = -Q_{-(n-1)}^{(3)} - 2Q_{-(n-2)}^{(3)} + Q_{-(n-3)}^{(3)} \tag{1.10}$$

and

$$E_{-n}^{(3)} = -E_{-(n-1)}^{(3)} - 2E_{-(n-2)}^{(3)} + E_{-(n-3)}^{(3)} \tag{1.11}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.9), (1.10) and (1.11) hold for all integer n.

In the rest of the paper, for easy writing, we drop the superscripts and write P_n, Q_n and E_n for $P_n^{(3)}, Q_n^{(3)}$ and $E_n^{(3)}$, respectively.

Note that P_n is the sequence A077939 in [1] associated with the expansion of $1/(1 - 2x - x^2 - x^3)$, Q_n is the sequence A276225 in [1] and Q_n is the sequence A077997 in [1].

Next, we present the first few values of the third-order Pell, third-order Pell-Lucas and modified third-order Pell numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
P_n	0	1	2	5	13	33	84	214	545	1388	3535	9003	22929	58396
P_{-n}	0	0	1	-1	-1	4	-3	-6	16	-7	-31	61	-6	-147
Q_n	3	2	6	17	42	107	273	695	1770	4508	11481	29240	74469	189659
Q_{-n}	3	-1	-3	8	-3	-16	30	-1	-75	107	42	-331	354	350
E_n	0	1	1	3	8	20	51	130	331	843	2147	5468	13926	35467
E_{-n}	0	-1	2	0	-5	7	3	-22	23	24	-92	67	141	-367

For all integers $n,$ third-order Pell, Pell-Lucas and modified Pell numbers (using initial conditions in (1.5)) can be expressed using Binet's formulas as

$$P_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$

and

$$Q_n = \alpha^n + \beta^n + \gamma^n,$$

and

$$E_n = \frac{(\alpha-1)\alpha^{n}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(\beta-1)\beta^{n}}{(\beta-\alpha)(\beta-\gamma)} + \frac{(\gamma-1)\gamma^{n}}{(\gamma-\alpha)(\gamma-\beta)},$$

respectively.

2 GENERATING FUNCTIONS

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 1. Suppose that $f_{V_n}(x) = \sum\limits_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized third-order Pell sequence $\{V_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} V_n x^n$ is given by

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2}{1 - 2x - x^2 - x^3}. \tag{2.1}$$

Proof. Using the definition of generalized third-order Pell numbers, and substracting $2x\sum_{n=0}^{\infty} V_n x^n$, $x^2\sum_{n=0}^{\infty} V_n x^n$ and $x^3\sum_{n=0}^{\infty} V_n x^n$ from $\sum_{n=0}^{\infty} V_n x^n$ we obtain

$$\begin{aligned}
(1-2x-x^2-x^3)\sum_{n=0}^{\infty} V_n x^n &= \sum_{n=0}^{\infty} V_n x^n - 2x\sum_{n=0}^{\infty} V_n x^n - x^2\sum_{n=0}^{\infty} V_n x^n - x^3\sum_{n=0}^{\infty} V_n x^n \\
&= \sum_{n=0}^{\infty} V_n x^n - 2\sum_{n=0}^{\infty} V_n x^{n+1} - \sum_{n=0}^{\infty} V_n x^{n+2} - \sum_{n=0}^{\infty} V_n x^{n+3} \\
&= \sum_{n=0}^{\infty} V_n x^n - 2\sum_{n=1}^{\infty} V_{n-1} x^{n} - \sum_{n=2}^{\infty} V_{n-2} x^{n} - \sum_{n=3}^{\infty} V_{n-3} x^{n} \\
&= (V_0 + V_1 x + V_2 x^2) - 2(V_0 x + V_1 x^2) - V_0 x^2 \\
&\quad + \sum_{n=3}^{\infty} (V_n - 2V_{n-1} - V_{n-2} - V_{n-3}) x^n \\
&= V_0 + V_1 x + V_2 x^2 - 2V_0 x - 2V_1 x^2 - V_0 x^2 \\
&= V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2.
\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2}{1 - 2x - x^2 - x^3}.$$

The previous Lemma gives the following results as particular examples.

Corollary 2. Generated functions of third-order Pell, Pell-Lucas and modified Pell numbers are

$$\sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-2x-x^2-x^3},$$

and

$$\sum_{n=0}^{\infty} Q_n x^n = \frac{3-4x-x^2}{1-2x-x^2-x^3},$$

and

$$\sum_{n=0}^{\infty} E_n x^n = \frac{x-x^2}{1-2x-x^2-x^3},$$

respectively.

3 OBTAINING BINET FORMULA FROM GENERATING FUNCTION

We next find Binet formula of generalized third order Pell numbers $\{V_n\}$ by the use of generating function for V_n.

Theorem 3. (Binet formula of generalized third order Pell numbers)

$$V_n = \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{3.1}$$

where

$$\begin{aligned} d_1 &= V_0\alpha^2 + (V_1-2V_0)\alpha + (V_2-2V_1-V_0), \\ d_2 &= V_0\beta^2 + (V_1-2V_0)\beta + (V_2-2V_1-V_0), \\ d_3 &= V_0\gamma^2 + (V_1-2V_0)\gamma + (V_2-2V_1-V_0). \end{aligned}$$

Proof. Let

$$h(x) = 1-2x-x^2-x^3.$$

Then for some α, β and γ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)$$

i.e.,

$$1-2x-x^2-x^3 = (1-\alpha x)(1-\beta x)(1-\gamma x) \tag{3.2}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}$, ve $\frac{1}{\gamma}$ are the roots of $h(x)$. This gives $\alpha, \beta,$ and γ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{2}{x} - \frac{1}{x^2} - \frac{1}{x^3} = 0.$$

This implies $x^3-2x^2-x-1=0$. Now, by (2.1) and (3.2), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1-2V_0)x + (V_2-2V_1-V_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)}.$$

Then we write

$$\frac{V_0 + (V_1-2V_0)x + (V_2-2V_1-V_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)} = \frac{A_1}{(1-\alpha x)} + \frac{A_2}{(1-\beta x)} + \frac{A_3}{(1-\gamma x)}. \tag{3.3}$$

So

$$V_0+(V_1-2V_0)x+(V_2-2V_1-V_0)x^2 = A_1(1-\beta x)(1-\gamma x)+A_2(1-\alpha x)(1-\gamma x)+A_3(1-\alpha x)(1-\beta x).$$

If we consider $x=\frac{1}{\alpha}$, we get $V_0+(V_1-2V_0)\frac{1}{\alpha}+(V_2-2V_1-V_0)\frac{1}{\alpha^2}=A_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})$. This gives

$$A_1=\frac{\alpha^2(V_0+(V_1-2V_0)\frac{1}{\alpha}+(V_2-2V_1-V_0)\frac{1}{\alpha^2})}{(\alpha-\beta)(\alpha-\gamma)}=\frac{V_0\alpha^2+(V_1-2V_0)\alpha+(V_2-2V_1-V_0)}{(\alpha-\beta)(\alpha-\gamma)}.$$

Similarly, we obtain

$$A_2=\frac{V_0\beta^2+(V_1-2V_0)\beta+(V_2-2V_1-V_0)}{(\beta-\alpha)(\beta-\gamma)}, A_3=\frac{V_0\gamma^2+(V_1-2V_0)\gamma+(V_2-2V_1-V_0)}{(\gamma-\alpha)(\gamma-\beta)}.$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty}V_nx^n=A_1(1-\alpha x)^{-1}+A_2(1-\beta x)^{-1}+A_3(1-\gamma x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty}V_nx^n=A_1\sum_{n=0}^{\infty}\alpha^nx^n+A_2\sum_{n=0}^{\infty}\beta^nx^n+A_3\sum_{n=0}^{\infty}\gamma^nx^n=\sum_{n=0}^{\infty}(A_1\alpha^n+A_2\beta^n+A_3\gamma^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n=A_1\alpha^n+A_2\beta^n+A_3\gamma^n$$

where

$$\begin{aligned}
A_1 &= \frac{V_0\alpha^2+(V_1-2V_0)\alpha+(V_2-2V_1-V_0)}{(\alpha-\beta)(\alpha-\gamma)},\\
A_2 &= \frac{V_0\beta^2+(V_1-2V_0)\beta+(V_2-2V_1-V_0)}{(\beta-\alpha)(\beta-\gamma)}\\
A_3 &= \frac{V_0\gamma^2+(V_1-2V_0)\gamma+(V_2-2V_1-V_0)}{(\gamma-\alpha)(\gamma-\beta)}.
\end{aligned}$$

and then we get (3.1).

Note that from (1.5) and (3.1) we have

$$\begin{aligned}
V_2-(\beta+\gamma)V_1+\beta\gamma V_0 &= V_0\alpha^2+(V_1-2V_0)\alpha+(V_2-2V_1-V_0),\\
V_2-(\alpha+\gamma)V_1+\alpha\gamma V_0 &= V_0\beta^2+(V_1-2V_0)\beta+(V_2-2V_1-V_0),\\
V_2-(\alpha+\beta)V_1+\alpha\beta V_0 &= V_0\gamma^2+(V_1-2V_0)\gamma+(V_2-2V_1-V_0).
\end{aligned}$$

Next, using Theorem 3, we present the Binet formulas of third-order Pell, Pell-Lucas and modified Pell sequences.

Corollary 4. Binet formulas of third-order Pell, Pell-Lucas and modified Pell sequences are

$$P_n=\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$

and

$$Q_n=\alpha^n+\beta^n+\gamma^n,$$

and

$$E_n = \frac{(\alpha-1)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(\beta-1)\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{(\gamma-1)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)},$$

respectively. Note that Binet formula of generalized third order Pell numbers can be represented as

$$V_n = \frac{\alpha d_1 \alpha^n}{2\alpha^2+2\alpha+3} + \frac{\beta d_2 \beta^n}{2\beta^2+2\beta+3} + \frac{\gamma d_3 \gamma^n}{2\gamma^2+2\gamma+3} \tag{3.4}$$

which can be derived from a result ((4.20) in page 25) of Hanusa [27]. When we compare (3.1) and (3.4), we see the following identities:

$$\begin{aligned} \frac{1}{(\alpha-\beta)(\alpha-\gamma)} &= \frac{\alpha}{2\alpha^2+2\alpha+3}, \\ \frac{1}{(\beta-\alpha)(\beta-\gamma)} &= \frac{\beta}{2\beta^2+2\beta+3}, \\ \frac{1}{(\gamma-\alpha)(\gamma-\beta)} &= \frac{\gamma}{2\gamma^2+2\gamma+3}. \end{aligned}$$

Using the above identities, we can give the Binet formulas of third-order Pell, Pell-Lucas and modified Pell sequences in the following form: Binet formulas of third-order Pell, Pell-Lucas and modified Pell sequences are

$$P_n = \frac{\alpha^{n+2}}{2\alpha^2+2\alpha+3} + \frac{\beta^{n+2}}{2\beta^2+2\beta+3} + \frac{\gamma^{n+2}}{2\gamma^2+2\gamma+3},$$

and

$$Q_n = \alpha^n + \beta^n + \gamma^n,$$

and

$$E_n = \frac{(\alpha-1)\alpha^{n+1}}{2\alpha^2+2\alpha+3} + \frac{(\beta-1)\beta^{n+1}}{2\beta^2+2\beta+3} + \frac{(\gamma-1)\gamma^{n+1}}{2\gamma^2+2\gamma+3}.$$

respectively.

We can find Binet formulas by using matrix method which is given in [12]. Take $k = i = 3$ in Corollary 3.1 in [12]. Let

$$\begin{aligned} \Lambda &= \begin{pmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{pmatrix}, \Lambda_1 = \begin{pmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{pmatrix}, \\ \Lambda_2 &= \begin{pmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{pmatrix}, \Lambda_3 = \begin{pmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{pmatrix}. \end{aligned}$$

Then the Binet formula for third-order Pell numbers is

$$\begin{aligned} P_n &= \frac{1}{\det(\Lambda)} \sum_{j=1}^{3} P_{4-j} \det(\Lambda_j) = \frac{1}{\Lambda}(P_3 \det(\Lambda_1) + P_2 \det(\Lambda_2) + P_1 \det(\Lambda_3)) \\ &= \frac{1}{\det(\Lambda)}(5\det(\Lambda_1) + 2\det(\Lambda_2) + \det(\Lambda_3)) \\ &= \left(5 \begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 2 \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{vmatrix} \right) / \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} \\ &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}. \end{aligned}$$

Similarly, we obtain the Binet formula for third-order Pell-Lucas and modified third-order Pell numbers as

$$\begin{aligned} Q_n &= \frac{1}{\Lambda}(Q_3 \det(\Lambda_1) + Q_2 \det(\Lambda_2) + Q_1 \det(\Lambda_3)) \\ &= \left(17\begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 6\begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} + 2\begin{vmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{vmatrix}\right) / \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} \\ &= \alpha^n + \beta^n + \gamma^n \end{aligned}$$

and

$$\begin{aligned} E_n &= \frac{1}{\Lambda}(E_3 \det(\Lambda_1) + E_2 \det(\Lambda_2) + E_1 \det(\Lambda_3)) \\ &= \left(3\begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{vmatrix}\right) / \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} \\ &= \frac{(\alpha-1)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(\beta-1)\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{(\gamma-1)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \end{aligned}$$

respectively.

4 SIMSON FORMULAS

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following Theorem gives generalization of this result to the generalized third-order Pell sequence $\{V_n\}_{n\geq 0}$.

Theorem 5. [Simson Formula of Generalized Third-Order Pell Numbers] For all integers n, we have

$$\begin{vmatrix} V_{n+2} & V_{n+1} & V_n \\ V_{n+1} & V_n & V_{n-1} \\ V_n & V_{n-1} & V_{n-2} \end{vmatrix} = \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}. \tag{4.1}$$

Proof. (4.1) is given in Soykan [28].

The previous Theorem gives the following results as particular examples.

Corollary 6. Simson formula of third-order Pell, Pell-Lucas and modified Pell numbers are given as

$$\begin{vmatrix} P_{n+2} & P_{n+1} & P_n \\ P_{n+1} & P_n & P_{n-1} \\ P_n & P_{n-1} & P_{n-2} \end{vmatrix} = -1,$$

and

$$\begin{vmatrix} Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+1} & Q_n & Q_{n-1} \\ Q_n & Q_{n-1} & Q_{n-2} \end{vmatrix} = -87,$$

and

$$\begin{vmatrix} E_{n+2} & E_{n+1} & E_n \\ E_{n+1} & E_n & E_{n-1} \\ E_n & E_{n-1} & E_{n-2} \end{vmatrix} = -3,$$

respectively.

5 SOME IDENTITIES

In this section, we obtain some identities of third order Pell, third order Pell-Lucas and modified third order Pell numbers. First, we can give a few basic relations between $\{P_n\}$ and $\{Q_n\}$.

Lemma 7. The following equalities are true:

$$\begin{aligned} Q_n &= 8P_{n+4} - 19P_{n+3} - 3P_{n+2}, \\ Q_n &= -3P_{n+3} + 5P_{n+2} + 8P_{n+1}, && (5.1) \\ Q_n &= -P_{n+2} + 5P_{n+1} - 3P_n, && (5.2) \\ Q_n &= 3P_{n+1} - 4P_n - P_{n-1}, && (5.3) \\ Q_n &= 2P_n + 2P_{n-1} + 3P_{n-2}, && (5.4) \end{aligned}$$

and

$$\begin{aligned} 87P_n &= 2Q_{n+4} - 18Q_{n+3} + 37Q_{n+2}, && (5.5) \\ 87P_n &= -14Q_{n+3} + 39Q_{n+2} + 2Q_{n+1}, && (5.6) \\ 87P_n &= 11Q_{n+2} - 12Q_{n+1} - 14Q_n, && (5.7) \\ 87P_n &= 10Q_{n+1} - 3Q_n + 11Q_{n-1}, && (5.8) \\ 87P_n &= 17Q_n + 21Q_{n-1} + 10Q_{n-2}, && (5.9) \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (5.1). To show (5.1), writing

$$Q_n = a \times P_{n+4} + b \times P_{n+3} + c \times P_{n+2}$$

and solving the system of equations

$$\begin{aligned} Q_0 &= a \times P_4 + b \times P_3 + c \times P_2 \\ Q_1 &= a \times P_5 + b \times P_4 + c \times P_3 \\ Q_2 &= a \times P_6 + b \times P_5 + c \times P_4 \end{aligned}$$

we find that $a = 8, b = -19, c = -3$. The other equalities can be proved similarly.

Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we present a few basic relations between $\{P_n\}$ and $\{E_n\}$.

Lemma 8. The following equalities are true:

$$\begin{aligned} E_n &= P_{n+3} - 5P_{n+2}, \\ E_n &= -P_{n+2} + 2P_{n+1} + 2P_n, \\ E_n &= P_n - P_{n-1}, \end{aligned}$$

and

$$\begin{aligned} 3P_n &= E_{n+3} - E_{n+2} - 2E_{n+1}, \\ 3P_n &= E_{n+2} - E_{n+1} + E_n, \\ 3P_n &= E_{n+1} + 2E_n + E_{n-1}. \end{aligned}$$

Thirdly, we give a few basic relations between $\{Q_n\}$ and $\{E_n\}$.

Lemma 9. The following equalities are true:

$$\begin{aligned} 3Q_n &= E_{n+3} - 4E_{n+2} + 10E_{n+1} \\ 3Q_n &= -2E_{n+2} + 11E_{n+1} + E_n \\ 3Q_n &= 7E_{n+1} - E_n - 2E_{n-1} \end{aligned}$$

and

$$\begin{aligned} 87E_n &= -16Q_{n+3} + 57Q_{n+2} - 35Q_{n+1} \\ 87E_n &= 25Q_{n+2} - 51Q_{n+1} - 16Q_n \\ 87E_n &= -Q_{n+1} + 9Q_n + 25Q_{n-1} \end{aligned}$$

We now present a few special identities for the modified third order Pell sequence $\{E_n\}$.

Theorem 10. (Catalan's identity) For all integers n and m, the following identity holds

$$\begin{aligned} E_{n+m}E_{n-m} - E_n^2 &= (P_{n+m} - P_{n+m-1})(P_{n-m} - P_{n-m-1}) - (P_n - P_{n-1})^2 \\ &= (P_n(P_m - P_{m+1}) + P_{n-1}(-P_m + P_{m-2}) + P_{n-2}(-P_m + P_{m-1})) \\ &\quad (P_n(P_{-m} - P_{1-m}) + P_{n-1}(-P_{-m} + P_{-m-2}) + P_{n-2}(-P_{-m} + P_{-m-1})) - (P_n - P_{n-1})^2 \end{aligned}$$

Proof. We use the identity

$$E_n = P_n - P_{n-1}$$

and the identity (7.6).

Note that for $m = 1$ in Catalan's identity, we get the Cassini identity for the modified third order Pell sequnce

Corollary 11. (Cassini's identity) For all integers numbers n and m, the following identity holds

$$E_{n+1}E_{n-1} - E_n^2 = (P_{n+1} - P_n)(P_{n-1} - P_{n-2}) - (P_n - P_{n-1})^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using $E_n = P_n - P_{n-1}$.The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of modified third order Pell sequence $\{E_n\}$.

Theorem 12. Let n and m be any integers. Then the following identities are true:

(a) (d'Ocagne's identity)

$$E_{m+1}E_n - E_mE_{n+1} = (P_{m+1} - P_m)(P_n - P_{n-1}) - (P_m - P_{m-1})(P_{n+1} - P_n).$$

(b) (Gelin-Cesàro's identity)

$$E_{n+2}E_{n+1}E_{n-1}E_{n-2} - E_n^4 = (P_{n+2} - P_{n+1})(P_{n+1} - P_n)(P_{n-1} - P_{n-2})(P_{n-2} - P_{n-3}) - (P_n - P_{n-1})^4$$

(c) (Melham's identity)

$$E_{n+1}E_{n+2}E_{n+6} - E_{n+3}^3 = (P_{n+1} - P_n)(P_{n+2} - P_{n+1})(P_{n+6} - P_{n+5}) - (P_{n+3} - P_{n+2})^3$$

Proof. Use the identity $E_n = P_n - P_{n-1}$.

6 LINEAR SUMS

The following Theorem presents some formulas of generalized third order Pell numbers.

Theorem 13. For $n \geq 0$ we have the following formulas:

(a) (Sum of the generalized third order Pell numbers)

$$\sum_{k=0}^{n} V_k = \frac{1}{3}\left(V_{n+3} - V_{n+2} - 2V_{n+1} - V_2 + V_1 + 2V_0\right)$$

(b)

$$\sum_{k=0}^{n} V_{2k} = \frac{1}{3}\left(V_{2n+1} + V_{2n} - V_1 + 2V_0\right)$$

(c)

$$\sum_{k=0}^{n} V_{2k+1} = \frac{1}{3}\left(V_{2n+2} + V_{2n+1} - V_2 + 2V_1\right).$$

Proof.

(a) Using the recurrence relation

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3}$$

i.e.

$$V_{n-3} = V_n - 2V_{n-1} - V_{n-2}$$

we obtain

$$\begin{aligned}
V_0 &= V_3 - 2V_2 - V_1 \\
V_1 &= V_4 - 2V_3 - V_2 \\
V_2 &= V_5 - 2V_4 - V_3 \\
V_3 &= V_6 - 2V_5 - V_4 \\
V_4 &= V_7 - 2V_6 - V_5 \\
&\vdots \\
V_{n-3} &= V_n - 2V_{n-1} - V_{n-2} \\
V_{n-2} &= V_{n+1} - 2V_n - V_{n-1} \\
V_{n-1} &= V_{n+2} - 2V_{n+1} - V_n \\
V_n &= V_{n+3} - 2V_{n+2} - V_{n+1}.
\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned}
\sum_{k=0}^{n} V_k &= (V_{n+3} + V_{n+2} + V_{n+1} - V_2 - V_1 - V_0 + \sum_{k=0}^{n} V_k) \\
&\quad -2(V_{n+2} + V_{n+1} - V_1 - V_0 + \sum_{k=0}^{n} V_k) - (V_{n+1} - V_0 + \sum_{k=0}^{n} V_k).
\end{aligned}$$

Then, solving the above equality we obtain

$$\sum_{k=0}^{n} V_k = \frac{1}{3}\left(V_{n+3} - V_{n+2} - 2V_{n+1} - V_2 + V_1 + 2V_0\right).$$

(b) and (c) Using the recurrence relation

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3}$$

i.e.

$$2V_{n-1} = V_n - V_{n-2} - V_{n-3}$$

we obtain

$$\begin{aligned}
2V_3 &= V_4 - V_2 - V_1 \\
2V_5 &= V_6 - V_4 - V_3 \\
2V_7 &= V_8 - V_6 - V_5 \\
2V_9 &= V_{10} - V_8 - V_7 \\
&\vdots \\
2V_{2n-1} &= V_{2n} - V_{2n-2} - V_{2n-3} \\
2V_{2n+1} &= V_{2n+2} - V_{2n} - V_{2n-1} \\
2V_{2n+3} &= V_{2n+4} - V_{2n+2} - V_{2n+1}.
\end{aligned}$$

Now, if we add the above equations by side by, we get

$$2(-V_1 + \sum_{k=0}^{n} V_{2k+1}) = (V_{2n+2} - V_2 - V_0 + \sum_{k=0}^{n} V_{2k}) - (-V_0 + \sum_{k=0}^{n} V_{2k}) - (-V_{2n+1} + \sum_{k=0}^{n} V_{2k+1}).$$

Similarly, using the recurrence relation

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3}$$

i.e.

$$2V_{n-1} = V_n - V_{n-2} - V_{n-3}$$

we write the following obvious equations;

$$\begin{aligned}
2V_2 &= V_3 - V_1 - V_0 \\
2V_4 &= V_5 - V_3 - V_2 \\
2V_6 &= V_7 - V_5 - V_4 \\
2V_8 &= V_9 - V_7 - V_6 \\
2V_{10} &= V_{11} - V_9 - V_8 \\
2V_{12} &= V_{13} - V_{11} - V_{10} \\
2V_{14} &= V_{15} - V_{13} - V_{12} \\
&\vdots \\
2V_{2n-2} &= V_{2n-1} - V_{2n-3} - V_{2n-4} \\
2V_{2n} &= V_{2n+1} - V_{2n-1} - V_{2n-2} \\
2V_{2n+2} &= V_{2n+3} - V_{2n+1} - V_{2n}.
\end{aligned}$$

Now, if we add the above equations by side by, we obtain

$$2(-V_0 + \sum_{k=0}^{n} V_{2k}) = (-V_1 + \sum_{k=0}^{n} V_{2k+1}) - (-V_{2n+1} + \sum_{k=0}^{n} V_{2k+1}) - (-V_{2n} + \sum_{k=0}^{n} V_{2k}).$$

Then, solving the following system

$$\begin{aligned} 2(-V_1+\sum_{k=0}^{n}V_{2k+1}) &= (V_{2n+2}-V_2-V_0+\sum_{k=0}^{n}V_{2k})-(-V_0+\sum_{k=0}^{n}V_{2k})-(-V_{2n+1}+\sum_{k=0}^{n}V_{2k+1}),\\ 2(-V_0+\sum_{k=0}^{n}V_{2k}) &= (-V_1+\sum_{k=0}^{n}V_{2k+1})-(-V_{2n+1}+\sum_{k=0}^{n}V_{2k+1})-(-V_{2n}+\sum_{k=0}^{n}V_{2k}),\end{aligned}$$

the required result of (b) and (c) follow.

As special cases of above Theorem, we have the following three Corollaries. First one presents some summing formulas of third order Pell numbers.

Corollary 14. For $n \geq 0$ we have the following formulas:

(a) (Sum of the third order Pell numbers)

$$\sum_{k=0}^{n} P_k = \frac{1}{3}\left(P_{n+3} - P_{n+2} - 2P_{n+1} - 1\right)$$

(b) $\sum_{k=0}^{n} P_{2k} = \frac{1}{3}\left(P_{2n+1} + P_{2n} - 1\right)$

(c) $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{3}\left(P_{2n+2} + P_{2n+1}\right).$

Second one presents some summing formulas of third order Pell-Lucas numbers.

Corollary 15. For $n \geq 0$ we have the following formulas:

(a) (Sum of the third order Pell-Lucas numbers)

$$\sum_{k=0}^{n} Q_k = \frac{1}{3}\left(Q_{n+3} - Q_{n+2} - 2Q_{n+1} + 2\right)$$

(b) $\sum_{k=0}^{n} Q_{2k} = \frac{1}{3}\left(Q_{2n+1} + Q_{2n} + 4\right)$

(c) $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{3}\left(Q_{2n+2} + Q_{2n+1} - 2\right).$

Third one presents some summing formulas of modified third order Pell numbers.

Corollary 16. For $n \geq 0$ we have the following formulas:

(a) (Sum of the modified third order Pell numbers)

$$\sum_{k=0}^{n} E_k = \frac{1}{3}\left(E_{n+3} - E_{n+2} - 2E_{n+1}\right)$$

(b) $\sum_{k=0}^{n} E_{2k} = \frac{1}{3}\left(E_{2n+1} + E_{2n} - 1\right)$

(c) $\sum_{k=0}^{n} E_{2k+1} = \frac{1}{3}\left(E_{2n+2} + E_{2n+1} + 1\right).$

7 MATRICES RELATED WITH GENERALIZED THIRD-ORDER PELL NUMBERS

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{7.1}$$

For matrix formulation (7.1), see [29]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det M = 1$. From (1.4) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix} \tag{7.2}$$

and from (7.1) (or using (7.2) and induction) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V = P$ in (7.2) we have

$$\begin{pmatrix} P_{n+2} \\ P_{n+1} \\ P_n \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{n+1} \\ P_n \\ P_{n-1} \end{pmatrix}. \tag{7.3}$$

We also define

$$B_n = \begin{pmatrix} P_{n+1} & P_n + P_{n-1} & P_n \\ P_n & P_{n-1} + P_{n-2} & P_{n-1} \\ P_{n-1} & P_{n-2} + P_{n-3} & P_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & V_n + V_{n-1} & V_n \\ V_n & V_{n-1} + V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-2} + V_{n-3} & V_{n-2} \end{pmatrix}$$

Theorem 17. For all integer $m, n \geq 0$, we have

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n.$

Proof.

(a) By expanding the vectors on the both sides of (7.3) to 3-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1} B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have

$$AC_{n-1} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_n & V_{n-1}+V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-2}+V_{n-3} & V_{n-2} \\ V_{n-2} & V_{n-3}+V_{n-4} & V_{n-3} \end{pmatrix}$$
$$= \begin{pmatrix} V_{n+1} & V_n+V_{n-1} & V_n \\ V_n & V_{n-1}+V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-2}+V_{n-3} & V_{n-2} \end{pmatrix} = C_n.$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and similarly

$$C_{n+m} = B_mC_n.$$

Some properties of matrix A^n can be given as

$$A^n = 2A^{n-1} + A^{n-2} + A^{n-3}$$

and

$$A^{n+m} = A^nA^m = A^mA^n$$

and

$$\det(A^n) = 1$$

for all integer m and n.

Theorem 18. For $m, n \geq 0$ we have

$$V_{n+m} = V_nP_{m+1} + V_{n-1}(P_m + P_{m-1}) + V_{n-2}P_m \quad (7.4)$$
$$= V_nP_{m+1} + (V_{n-1} + V_{n-2})P_m + V_{n-1}P_{m-1}. \quad (7.5)$$

Proof. From the equation $C_{n+m} = C_nB_m = B_mC_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and C_nB_m. This completes the proof.

Remark 19. By induction, it can be proved that for all integers $m, n \leq 0$, (7.4) holds. So for all integers m, n, (7.4) is true.

Corollary 20. For all integers m, n, we have

$$P_{n+m} = P_nP_{m+1} + P_{n-1}(P_m + P_{m-1}) + P_{n-2}P_m \quad (7.6)$$
$$Q_{n+m} = Q_nP_{m+1} + Q_{n-1}(P_m + P_{m-1}) + Q_{n-2}P_m \quad (7.7)$$
$$E_{n+m} = E_nP_{m+1} + E_{n-1}(P_m + P_{m-1}) + E_{n-2}P_m \quad (7.8)$$

8 CONCLUSION

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. We introduce the generalized third order Pell sequences and we present Binet's formulas, generating functions, Simson formulas, the summation formulas, some identities and matrices for these sequences.

COMPETING INTERESTS

Authors has declared that no competing interests exist.

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