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Third Derivative Block Method for the Direct Solution of Second-order Ordinary Differential Equations Using Two-step Hybrid Block Method of Order Ten

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Authors' contributions

This work was carried out in collaboration between all authors. Author DR designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors YS and SJ managed the analyses of the study. Author TYK managed the literature searches. All authors read and approved the final manuscript.

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Abstract

This research aim at developing two-step block third derivative method with two-offstep points for a direct solution of stiff second order ODEs. Power series approximation method was adopted as basis function using interpolation and collocation techniques at two selected grid points. We investigate some basic properties of our method and found it to converge. Stability region plotted shows that the method is A-stable and can handle stiff problems. The method was then tested on some highly stiff problems and numerical results presented side by side in tables' shows that our method performs efficiently better on stiff and highly stiff problems as compared with an existing method.

Keywords: Two-step; hybrid; block method; stiff.

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1 Introduction

Considering the numerical solution to general second Order initial value problem in the form

$$y'' = f(x, y(x), y'(x)) , \quad y(x_0) = y_0 , \; y'(x_0) = y'_0 \tag{1}$$

where $x \in [a,b]$ and $y \in \Re$ and $y^r, r = 0,1$ denote the derivative of the dependent variable y with respect to the independent variable x.

Different approaches in solving (1) have been proposed by many author ranging from reducing to first order to predictor-corrector methods and then to hybrid methods. However, some drawbacks were noticed in this approaches which result in more functions to evaluate partime and as the step length is high its lead to computational burden as mentioned in [1-3]. we shall employ the hybrid block method due to it advantages of incorporating function at offgrid points that gives the oppurtunity of restricting the Dahlquist zero barrier stability.

Stiff differential equations were first encountered in the study of the motion of springs varying stiffness, from which the problem derives its name. Stiffness occurs when some components of the solution decay much more rapidly than others.

Collocation and interpolation of the power series approximation was adopted by some authors to generate continuous linear multistep method, few among them are [4,5,6,7,8,9]. Independent solution at selected grid points was generated using block method without overlapping. Block method possess the properties of Runge Kutta method for being self-starting and does not require starting value, it is easier and less expensive in terms of number of function evaluation as compared to reduction to first Order and predictor corrector method. Few among the authors that proposed block method are [10,11,12,8,13,9,14,15].

Many authors proposes various methods for solving (1), some adopt power series, chebyshev, and langrange polynomials as their basic functions. In this paper, we develop an order ten two-step hybrid block third derivative method that gives better stability condition by using power series as our basis function.

2 Derivation of the Method

We consider a power series approximate solution of the form

$$y(x) = \sum_{j=0}^{2s+r-1} a_i \left(\frac{x - x_n}{h} \right)^j \tag{2}$$

where $r = 2$ and $s = 5$ are the numbers of interpolation and collocation points respectively, is considered to be a solution to (1).

The second and third derivative of (2) gives

$$\begin{aligned} y''(x) &= \sum_{j=2}^{2s+r-1} \frac{a_j j!}{h^2 (j-2)} \left(\frac{x - x_n}{h} \right)^{j-2} \\ &= f(x, y, y'), \end{aligned} \tag{3}$$

$$y'''(x) = \sum_{j=3}^{2s+r-1} \frac{a_j j!}{h^3 (j-3)} \left(\frac{x - x_n}{h} \right)^{j-3}$$
$$= g(x, y, y'),$$

Substituting (3) into (1) gives

$$f(x, y, y'') = \sum_{j=2}^{2s+r-1} \frac{a_j j!}{h^2 (j-2)} \left(\frac{x - x_n}{h} \right)^{j-2} + \sum_{j=3}^{2s+r-1} \frac{a_j j!}{h^3 (j-3)} \left(\frac{x - x_n}{h} \right)^{j-3} \tag{4}$$

Collocating (4) at all points $x_{n+s}, s = 0\left(\frac{1}{2}\right)2$ and Interpolating Equation (2) at $x_{n+r}, r = 0, \frac{1}{2}$, gives a system of non linear equation of the form

$$AX = U \tag{5}$$

where

$$A = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}]^T,$$

$$U = \left[y_n, y_{n+\frac{1}{2}}, f_n, f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}, g_n, g_{n+\frac{1}{2}}, g_{n+1}, g_{n+\frac{3}{2}}, g_{n+2} \right]^T,$$

and

$$X = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \frac{1}{128} & \frac{1}{256} & \frac{1}{512} & \frac{1}{1024} & \frac{1}{2048} \\
0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{h^2} & \frac{3}{h^2} & \frac{3}{h^2} & \frac{5}{2h^2} & \frac{15}{8h^2} & \frac{21}{16h^2} & \frac{7}{8h^2} & \frac{9}{16h^2} & \frac{45}{128h^2} & \frac{55}{256h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} & \frac{42}{h^2} & \frac{56}{h^2} & \frac{72}{h^2} & \frac{90}{h^2} & \frac{110}{h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{9}{h^2} & \frac{9}{h^2} & \frac{15}{2h^2} & \frac{45}{8h^2} & \frac{63}{16h^2} & \frac{21}{8h^2} & \frac{27}{16h^2} & \frac{135}{128h^2} & \frac{165}{256h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{12}{h^2} & \frac{24}{h^2} & \frac{40}{h^2} & \frac{60}{h^2} & \frac{84}{h^2} & \frac{112}{h^2} & \frac{144}{h^2} & \frac{180}{h^2} & \frac{220}{h^2} \\
0 & 0 & 0 & \frac{6}{h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{6}{h^3} & \frac{12}{h^3} & \frac{15}{h^3} & \frac{15}{h^3} & \frac{105}{8h^3} & \frac{21}{2h^3} & \frac{63}{8h^3} & \frac{45}{8h^3} & \frac{495}{128h^3} \\
0 & 0 & 0 & \frac{6}{h^3} & \frac{24}{h^3} & \frac{60}{h^3} & \frac{120}{h^3} & \frac{210}{h^3} & \frac{336}{h^3} & \frac{504}{h^3} & \frac{720}{h^3} & \frac{990}{h^3} \\
0 & 0 & 0 & \frac{6}{h^3} & \frac{36}{h^3} & \frac{45}{h^3} & \frac{45}{h^3} & \frac{155}{4h^3} & \frac{63}{2h^3} & \frac{189}{8h^3} & \frac{135}{8h^3} & \frac{1485}{128h^3} \\
0 & 0 & 0 & \frac{6}{h^3} & \frac{48}{h^3} & \frac{120}{h^3} & \frac{240}{h^3} & \frac{420}{h^3} & \frac{672}{h^3} & \frac{1008}{h^3} & \frac{1440}{h^3} & \frac{1980}{h^3}
\end{bmatrix}$$

Solving (5) for $a_i's$ employing Gaussian elimination method, gives a continuous hybrid linear multistep method of the form

$$y(x)=\sum_{j=0,\frac{1}{2}} \alpha_j y_{n+j} + h^2\left[\sum_{j=0}^{2}\beta_j f_{n+j} + \beta_k f_{n+k}\right] + h^3\left[\sum_{j=0}^{2}\gamma_j g_{n+j} + \gamma_k g_{n+k}\right],\ k=\frac{1}{2},\frac{3}{2} \tag{6}$$

Differentiating (6) once yields

$$p'(x)=\frac{1}{h}\sum_{j=0,\frac{1}{2}} \alpha_j y_{n+j} + h\left[\sum_{j=\frac{1}{2},\frac{3}{2}}\beta_j f_{n+j} + \sum_{j=0}^{2}\beta_j f_{n+j}\right] + h^2\left[\sum_{j=\frac{1}{2},\frac{3}{2}}\gamma_j g_{n+j} + \sum_{j=0}^{2}\gamma_j g_{n+j}\right] \tag{7}$$

where

$$a_0=\left(1-\frac{2(x-x_n)}{h}\right),\qquad a_{\frac{1}{2}}=\left(=\frac{2(x-x_n)}{h}\right)$$

$$\beta_0=\frac{10}{297}\frac{(x-x_n)^{11}}{h^9}-\frac{494}{1215}\frac{(x-x_n)^{10}}{h^8}+\frac{2065}{972}\frac{(x-x_n)^9}{h^7}-\frac{4745}{756}\frac{(x-x_n)^8}{h^6}+\frac{52025}{4536}\frac{(x-x_n)^7}{h^5}-\frac{42931}{3240}\frac{(x-x_n)^6}{h^4}$$
$$+\frac{4031}{432}\frac{(x-x_n)^5}{h^3}-\frac{485}{144}\frac{(x-x_n)^4}{h^2}+\frac{1}{2}(x-x_n)^2-\frac{2602339}{19160064}(x-x_n)h$$

$$\beta_{\frac{1}{2}}=\frac{64}{297}\frac{(x-x_n)^{11}}{h^9}-\frac{2944}{1215}\frac{(x-x_n)^{10}}{h^8}+\frac{2804}{243}\frac{(x-x_n)^9}{h^7}-\frac{5692}{189}\frac{(x-x_n)^8}{h^6}+\frac{26288}{567}\frac{(x-x_n)^7}{h^5}-\frac{16912}{405}\frac{(x-x_n)^6}{h^4}$$
$$+\frac{896}{45}\frac{(x-x_n)^5}{h^3}-\frac{32}{9}\frac{(x-x_n)^4}{h^2}-\frac{148231}{5987520}(x-x_n)h$$

$$\beta_1=\frac{8}{45}\frac{(x-x_n)^{10}}{h^8}-\frac{16}{9}\frac{(x-x_n)^9}{h^7}+\frac{51}{7}\frac{(x-x_n)^8}{h^6}-\frac{328}{21}\frac{(x-x_n)^7}{h^5}+\frac{553}{30}\frac{(x-x_n)^6}{h^4}-\frac{57}{5}\frac{(x-x_n)^5}{h^3}+\frac{3}{9}\frac{(x-x_n)^4}{h^2}$$
$$-\frac{1807}{40320}(x-x_n)h$$

$$\beta_{\frac{3}{2}}=-\frac{64}{297}\frac{(x-x_n)^{11}}{h^9}+\frac{2816}{1215}\frac{(x-x_n)^{10}}{h^8}-\frac{2548}{243}\frac{(x-x_n)^9}{h^7}+\frac{4892}{189}\frac{(x-x_n)^8}{h^6}-\frac{21424}{567}\frac{(x-x_n)^7}{h^5}+\frac{13328}{405}\frac{(x-x_n)^6}{h^4}$$
$$-\frac{2176}{135}\frac{(x-x_n)^5}{h^3}+\frac{32}{9}\frac{(x-x_n)^4}{h^2}-\frac{243193}{5987520}(x-x_n)h$$

$$\beta_2 = -\frac{10}{297}\frac{(x-x_n)^{11}}{h^9} + \frac{406}{1215}\frac{(x-x_n)^{10}}{h^8} - \frac{1361}{972}\frac{(x-x_n)^9}{h^7} + \frac{2437}{756}\frac{(x-x_n)^8}{h^6} - \frac{20089}{4536}\frac{(x-x_n)^7}{h^5} + \frac{11879}{3240}\frac{(x-x_n)^6}{h^4} - \frac{1241}{720}\frac{(x-x_n)^5}{h^3} + \frac{53}{144}\frac{(x-x_n)^4}{h^2} - \frac{382169}{95800320}(x-x_n)h$$

$$\gamma_0 = \frac{2}{495}\frac{(x-x_n)^{11}}{h^8} - \frac{4}{81}\frac{(x-x_n)^{10}}{h^7} + \frac{85}{324}\frac{(x-x_n)^9}{h^6} - \frac{50}{63}\frac{(x-x_n)^8}{h^5} + \frac{2273}{1512}\frac{(x-x_n)^7}{h^4} - \frac{199}{108}\frac{(x-x_n)^6}{h^3} + \frac{209}{144}\frac{(x-x_n)^5}{h^2} - \frac{25}{36}\frac{(x-x_n)^4}{h} + \frac{1}{6}(x-x_n)^3 - \frac{28343}{4561920}(x-x_n)h^2$$

$$\gamma_{\frac{1}{2}} = \frac{32}{495}\frac{(x-x_n)^{11}}{h^8} - \frac{304}{405}\frac{(x-x_n)^{10}}{h^7} + \frac{302}{81}\frac{(x-x_n)^9}{h^6} - \frac{649}{63}\frac{(x-x_n)^8}{h^5} + \frac{464}{27}\frac{(x-x_n)^7}{h^4} - \frac{2356}{135}\frac{(x-x_n)^6}{h^3} + \frac{152}{15}\frac{(x-x_n)^5}{h^2} - \frac{8}{3}\frac{(x-x_n)^4}{h} + \frac{551}{12474}(x-x_n)h^2$$

$$\gamma_1 = \frac{8}{55}\frac{(x-x_n)^{11}}{h^8} - \frac{8}{5}\frac{(x-x_n)^{10}}{h^7} + \frac{67}{9}\frac{(x-x_n)^9}{h^6} - 19\frac{(x-x_n)^8}{h^5} + \frac{403}{14}\frac{(x-x_n)^7}{h^4} - \frac{781}{30}\frac{(x-x_n)^6}{h^3} + \frac{66}{5}\frac{(x-x_n)^5}{h^2} - 3\frac{(x-x_n)^4}{h} + \frac{32027}{887040}(x-x_n)h^2$$

$$\gamma_{\frac{3}{2}} = \frac{32}{495}\frac{(x-x_n)^{11}}{h^8} - \frac{272}{405}\frac{(x-x_n)^{10}}{h^7} + \frac{238}{81}\frac{(x-x_n)^9}{h^6} - \frac{443}{63}\frac{(x-x_n)^8}{h^5} + \frac{1888}{189}\frac{(x-x_n)^7}{h^4} - \frac{1148}{135}\frac{(x-x_n)^6}{h^3} + \frac{184}{45}\frac{(x-x_n)^5}{h^2} - \frac{8}{9}\frac{(x-x_n)^4}{h} + \frac{3959}{399168}(x-x_n)h^2$$

$$\gamma_2 = \frac{2}{495}\frac{(x-x_n)^{11}}{h^8} - \frac{16}{405}\frac{(x-x_n)^{10}}{h^7} + \frac{53}{324}\frac{(x-x_n)^9}{h^6} - \frac{47}{126}\frac{(x-x_n)^8}{h^5} + \frac{769}{1512}\frac{(x-x_n)^7}{h^4} - \frac{113}{270}\frac{(x-x_n)^6}{h^3} + \frac{47}{240}\frac{(x-x_n)^5}{h^2} - \frac{1}{24}\frac{(x-x_n)^4}{h} + \frac{14339}{31933440}(x-x_n)h^2$$

Equation (6) is evaluated at the non-interpolating points $\left\{x_{n+1}, x_{n+\frac{3}{2}}, x_{n+2}\right\}$ and (7) at all points $\left\{x_n, x_{n+\frac{1}{2}}, x_{n+1}, x_{n+\frac{3}{2}}, x_{n+2}\right\}$, produces the following general equations in block form

$$AY_L = BR_1 + CR_2 + DR_3 + ER_4 + GR_5 \tag{8}$$

where

$$A=\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ \frac{-2}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \frac{-2}{h} & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ \frac{-2}{h} & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ \frac{-2}{h} & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ \frac{-2}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},\quad Y=\begin{bmatrix} y_{n+\frac{1}{2}}\\ y_{n+1}\\ y_{n+\frac{3}{2}}\\ y_{n+2}\\ y'_{n+\frac{1}{2}}\\ y'_{n+1}\\ y'_{n+\frac{3}{2}}\\ y'_{n+2} \end{bmatrix},\quad B=\begin{bmatrix} 1 & 0\\ 2 & 0\\ 3 & 0\\ \frac{2}{h} & 1\\ \frac{2}{h} & 0\\ \frac{2}{h} & 0\\ \frac{2}{h} & 0\\ \frac{2}{h} & 0 \end{bmatrix},\ R_1=\begin{bmatrix} y_n\\ y'_n \end{bmatrix},$$

$$C=\begin{bmatrix} \frac{37111h^2}{1741824}\\ \frac{12679h^2}{290304}\\ \frac{19709h^2}{290304}\\ -\frac{2602339h}{19160064}\\ \frac{1961683h}{47900160}\\ \frac{1403419h}{31933440}\\ \frac{2182739h}{47900160}\\ \frac{5021969h}{95800320} \end{bmatrix},\ R_2=\begin{bmatrix} f_n \end{bmatrix}\ E=\begin{bmatrix} \frac{4253h^3}{2903040}\\ \frac{23h^3}{7560}\\ \frac{467h^3}{96768}\\ -\frac{28343h^2}{4561920}\\ \frac{551h^2}{199584}\\ \frac{3629h^2}{1182720}\\ \frac{3239h^2}{997920}\\ \frac{25651h^2}{6386688} \end{bmatrix},\ R_4=\begin{bmatrix} g_n \end{bmatrix},$$

$$D=\begin{bmatrix} \frac{16463h^2}{108864} & \frac{445h^2}{8064} & \frac{2225h^2}{108864} & \frac{3217h^2}{1741824}\\ \frac{6149h^2}{18144} & \frac{767h^2}{2688} & \frac{1403h^2}{18144} & \frac{1381h^2}{290304}\\ \frac{9925h^2}{18144} & \frac{765h^2}{1344} & \frac{5179h^2}{18144} & \frac{8411h^2}{290304}\\ -\frac{148231h}{5987520} & -\frac{1807h}{40320} & -\frac{243193h}{5987520} & -\frac{382169h}{95800320}\\ \frac{83485h}{598752} & \frac{1489h}{40320} & \frac{89279h}{2993760} & \frac{137161h}{47900160}\\ \frac{26921h}{73920} & \frac{767h}{2688} & \frac{103853h}{1995840} & \frac{5303h}{1182720}\\ \frac{1156801h}{2993760} & \frac{21521h}{40320} & \frac{165731h}{598752} & \frac{358217h}{47900160}\\ \frac{2735353h}{5987520} & \frac{24817h}{40320} & \frac{2640391h}{5987520} & \frac{3530299h}{19160064} \end{bmatrix},\ R_3=\begin{bmatrix} f_{n+\frac{1}{2}}\\ f_{n+1}\\ f_{n+\frac{3}{2}}\\ f_{n+2} \end{bmatrix}$$

$$G=\begin{bmatrix} -\frac{17h^3}{1134} & -\frac{83h^3}{3840} & -\frac{871h^3}{181440} & -\frac{599h^3}{2903040}\\ -\frac{3053h^3}{120960} & -\frac{83h^3}{1920} & -\frac{347h^3}{24192} & -\frac{127h^3}{241920}\\ -\frac{29h^3}{945} & -\frac{83h^3}{1920} & -\frac{269h^3}{30240} & -\frac{1117h^3}{483840}\\ \frac{551h^2}{12474} & \frac{32027h^2}{887040} & \frac{3959h^2}{399168} & \frac{14339h^2}{31933440}\\ -\frac{23851h^2}{570240} & -\frac{12053h^2}{443520} & -\frac{5755h^2}{798336} & -\frac{2567h^2}{7983360}\\ -\frac{95h^2}{4158} & -\frac{56677h^2}{887040} & -\frac{7951h^2}{665280} & -\frac{5311h^2}{10644480}\\ -\frac{72269h^2}{3991680} & -\frac{12053h^2}{443520} & -\frac{123463h^2}{3991680} & -\frac{6439h^2}{7983360}\\ -\frac{61h^2}{62370} & \frac{32027h^2}{887040} & \frac{15701h^2}{285120} & -\frac{312317h^2}{31933440} \end{bmatrix},\ R_5=\begin{bmatrix} g_{n+\frac{1}{2}}\\ g_{n+1}\\ g_{n+\frac{3}{2}}\\ g_{n+2} \end{bmatrix}$$

Multipling equation (8) by the inverse of (A) gives the hybrid block formula of the form

$$IY_L=\bar{B}R_1+\bar{C}R_2+\bar{D}R_3+\bar{E}R_4+\bar{G}R_5 \tag{9}$$

$$I=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},\quad \bar{B}=\begin{bmatrix} 1 & \frac{h}{2}\\ 1 & h\\ 1 & \frac{3h}{2}\\ 1 & 2h\\ 0 & 1\\ 0 & 1\\ 0 & 1\\ 0 & 1 \end{bmatrix},\ \bar{C}=\begin{bmatrix} \frac{2602339h^2}{38320128}\\ \frac{140h^2}{891}\\ \frac{39015h^2}{157696}\\ \frac{6353h^2}{18711}\\ \frac{1539551h}{8709120}\\ \frac{24463h}{136080}\\ \frac{6501h}{35840}\\ \frac{1601h}{8505} \end{bmatrix},$$

$$\bar{D}=\begin{bmatrix} \frac{148231h^2}{11975040} & \frac{1807h^2}{80640} & \frac{243193h^2}{11975040} & \frac{382169h^2}{191600640}\\ \frac{784h^2}{4455} & \frac{h^2}{10} & \frac{272h^2}{4455} & \frac{26h^2}{4455}\\ \frac{18531h^2}{49280} & \frac{3159h^2}{8960} & \frac{6813h^2}{49280} & \frac{8469h^2}{788480}\\ \frac{159552h^2}{48125} & \frac{208h^2}{315} & \frac{34304h^2}{93555} & \frac{3457h^2}{93555}\\ \frac{89371h}{544320} & \frac{103h}{1260} & \frac{38341h}{544320} & \frac{59681h}{8709120}\\ \frac{3308h}{8505} & \frac{104h}{315} & \frac{788h}{8505} & \frac{1153h}{136080}\\ \frac{921h}{2240} & \frac{81h}{140} & \frac{711h}{2240} & \frac{411h}{35840}\\ \frac{4096h}{8505} & \frac{208h}{315} & \frac{4096h}{8505} & \frac{1601h}{8505} \end{bmatrix}$$

$$\bar{E} = \begin{bmatrix} \frac{28343h^3}{9123840} \\ \frac{1277h^3}{166320} \\ \frac{9747h^3}{788480} \\ \frac{538h^3}{31185} \\ \frac{26051h^2}{2903040} \\ \frac{421h^2}{45360} \\ \frac{339h^2}{35840} \\ \frac{29h^2}{2835} \end{bmatrix}, \bar{G} = \begin{bmatrix} \frac{-551h^3}{24948} & \frac{-32027h^3}{1774080} & \frac{-3959h^3}{798336} & \frac{-14339h^3}{63866880} \\ \frac{-41h^3}{693} & \frac{-40h^3}{693} & \frac{-17h^3}{1155} & \frac{-109h^3}{166320} \\ \frac{-4509h^3}{49280} & \frac{-19197h^3}{197120} & \frac{-9h^3}{308} & \frac{-27h^3}{22528} \\ \frac{-3712h^3}{31185} & \frac{-80h^3}{693} & \frac{-128h^3}{4455} & \frac{-20h^3}{6237} \\ \frac{-31207h^2}{362880} & \frac{-81h^2}{1280} & \frac{-1243h^2}{72576} & \frac{-2237h^2}{2903040} \\ \frac{-38h^2}{567} & \frac{-h^2}{10} & \frac{-62h^2}{2835} & \frac{-43h^2}{45360} \\ \frac{-279h^2}{4480} & \frac{-81h^2}{1280} & \frac{-183h^2}{4480} & \frac{-9h^2}{7168} \\ \frac{-128h^2}{2835} & \frac{128h^2}{2835} & 0 & \frac{-29h^2}{2835} \end{bmatrix}$$

3 Analysis of Basic Properties of the Method

3.1 Order of the block

Let the linear operator associated with the block (9) be defined as,

$$L\{y(x);h\} = IY_L - \bar{B}R_1 - \bar{C}R_2 - \bar{D}R_3 - \bar{E}R_4 - \bar{G}R_5 \tag{10}$$

Expanding (10) using Taylor series and comparing the coefficients in h gives

$$L\{y(x);h\} = c_0 y(x) + c_1 y'(x) + c_2 h^2 y''(x) + \ldots + c_p h^p y^p(x) + c_{p+1} h^{p+1} y^{p+1}(x) + c_{p+2} h^{p+2} y^{p+2}(x) \ldots \tag{11}$$

Definition 1: The linear operator and the associated continuous linear multistep method (6) are said to of order p if $c_0 = c_1 = c_2 = \ldots = c_p = c_{p+1} = 0,\ c_{p+2} \neq 0, c_{p+2}$ is called the error constant and the local truncation error is given by

$$t_{n+k} = c_{p+2} h^{(p+2)} y^{(p+2)}(x_n) + o(h^{p+3}) \tag{12}$$

For our method

$$
\begin{bmatrix}
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}h\right)^j}{j!}y_n^{(j)}-y_n-\frac{1}{2}hy_n'-\frac{2602339}{38320128}h^2y_n''-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\begin{array}{l}\frac{148231}{11975040}\left(\frac{1}{2}\right)^j+\frac{1807}{80640}(1)^j\\+\frac{243193}{11975040}\left(\frac{3}{2}\right)^j+\frac{382169}{191600640}(2)^j\end{array}\right]-\frac{28343}{9123840}h^3y_n'''+\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\left[\begin{array}{l}\frac{551}{24948}\left(\frac{1}{2}\right)^j+\frac{32027}{1774080}(1)^j\\+\frac{3959}{798336}\left(\frac{3}{2}\right)^j+\frac{14339}{63866880}(2)^j\end{array}\right]\\
\sum_{j=0}^{\infty}\frac{(1h)^j}{j!}y_n^{(j)}-y_n-hy_n'-\frac{140}{891}h^2y_n''-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\begin{array}{l}\frac{784}{4455}\left(\frac{1}{2}\right)^j+\frac{1}{10}(1)^j\\+\frac{272}{4455}\left(\frac{3}{2}\right)^j+\frac{26}{4455}(2)^j\end{array}\right]-\frac{1277}{166320}h^3y_n'''+\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\left[\begin{array}{l}\frac{41}{693}\left(\frac{1}{2}\right)^j+\frac{40}{693}(1)^j\\+\frac{17}{1155}\left(\frac{1}{2}\right)^j+\frac{109}{166320}(2)^j\end{array}\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{3}{2}h\right)^j}{j!}y_n^{(j)}-y_n-\frac{3}{2}hy_n'-\frac{39015}{157696}h^2y_n''-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\begin{array}{l}\frac{18531}{49280}\left(\frac{1}{2}\right)^j+\frac{3159}{8960}(1)^j\\+\frac{6813}{49280}\left(\frac{3}{2}\right)^j+\frac{8469}{788480}(2)^j\end{array}\right]-\frac{9747}{788480}h^3y_n'''+\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\left[\begin{array}{l}\frac{4509}{49280}\left(\frac{1}{2}\right)^j+\frac{19197}{197120}(1)^j\\+\frac{9}{308}\left(\frac{3}{2}\right)^j+\frac{27}{22528}(2)^j\end{array}\right]\\
\sum_{j=0}^{\infty}\frac{(2h)^j}{j!}y_n^{(j)}-2hy_n'-\frac{6353}{18711}h^2y_n''-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2}\left[\begin{array}{l}\frac{55808}{93555}\left(\frac{1}{2}\right)^j+\frac{208}{315}(1)^j\\+\frac{34304}{93555}\left(\frac{3}{2}\right)^j+\frac{3457}{93555}(2)^j\end{array}\right]-\frac{538}{31183}h^3y_n'''+\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3}\left[\begin{array}{l}\frac{3712}{31185}\left(\frac{1}{2}\right)^j+\frac{80}{693}(1)^j\\+\frac{128}{4455}\left(\frac{31}{2}\right)^j+\frac{20}{6237}(2)^j\end{array}\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}h\right)^j}{j!}y'^{(j)}_n-y'_n-\frac{1539551}{8709120}hy_n''-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+2}\left[\begin{array}{l}\frac{89371}{544320}\left(\frac{1}{2}\right)^j+\frac{103}{1260}(1)^j\\+\frac{38341}{544320}\left(\frac{3}{2}\right)^j+\frac{59681}{8709120}(2)^j\end{array}\right]-\frac{26051}{2903040}h^2y_n'''+\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+3}\left[\begin{array}{l}\frac{31207}{362880}\left(\frac{1}{2}\right)^j+\frac{81}{1280}(1)^j\\+\frac{1243}{72576}\left(\frac{3}{2}\right)^j+\frac{2237}{2903040}(2)^j\end{array}\right]\\
\sum_{j=0}^{\infty}\frac{(1h)^j}{j!}y'^{(j)}_n-y'_n-\frac{24463}{136080}hy_n''-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+2}\left[\begin{array}{l}\frac{3308}{8505}\left(\frac{1}{2}\right)^j+\frac{104}{315}(1)^j\\+\frac{788}{8505}\left(\frac{3}{2}\right)^j+\frac{1153}{136080}(2)^j\end{array}\right]-\frac{421}{45360}h^2y_n'''+\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+3}\left[\begin{array}{l}\frac{38}{567}\left(\frac{1}{2}\right)^j+\frac{1}{10}(1)^j\\+\frac{62}{2835}\left(\frac{3}{2}\right)^j+\frac{43}{45360}(2)^j\end{array}\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{3}{2}h\right)^j}{j!}y'^{(j)}_n-y'_n-\frac{6501}{35840}hy_n''-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+2}\left[\begin{array}{l}\frac{921}{2240}\left(\frac{1}{2}\right)^j+\frac{81}{140}(1)^j\\+\frac{711}{2240}\left(\frac{3}{2}\right)^j+\frac{411}{35840}(2)^j\end{array}\right]-\frac{339}{35840}h^2y_n'''+\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+3}\left[\begin{array}{l}\frac{279}{4480}\left(\frac{1}{2}\right)^j+\frac{81}{1280}(1)^j\\+\frac{183}{4480}\left(\frac{3}{2}\right)^j+\frac{9}{7168}(2)^j\end{array}\right]\\
\sum_{j=0}^{\infty}\frac{(2h)^j}{j!}y'^{(j)}_n-y'_n-\frac{1601}{8505}hy_n''-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+2}\left[\begin{array}{l}\frac{4096}{8505}\left(\frac{1}{2}\right)^j+\frac{208}{315}(1)^j\\+\frac{4096}{8505}\left(\frac{3}{2}\right)^j+\frac{1601}{8505}(2)^j\end{array}\right]-\frac{29}{2835}h^2y_n'''-\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+3}\left[\begin{array}{l}\frac{128}{2835}\left(\frac{1}{2}\right)^j+0(1)^j\\-\frac{128}{2835}\left(\frac{3}{2}\right)^j+\frac{29}{2835}(2)^j\end{array}\right]
\end{bmatrix}=0
$$

Comparing the coefficient of h gives $C_0=C_1=C_2=C_3=...=C_{10}=0$ and

$$C_{11}=\left[\frac{20869}{430080},\frac{11}{30720},\frac{141}{143360},\frac{1}{5376},\frac{551}{73920},\frac{1}{7392},\frac{1}{24640},\frac{4131}{11200}\right]^T$$

Therefore, our hybrid block method is of uniform order ten.

3.2 Zero stability of our method

Definition: A block method is said to be zero-stable if as $h\to 0$, the root $z_i, i=1(1)k$ of the first characteristic polynomial $\rho(z)=0$ that is $\rho(z)=\det\left[\sum_{j=0}^{k}A^{(i)}z^{k-i}\right]=0$ Satisfies $|z_i|\le 1$ and for those roots with $|z_i|$=1, multiplicity must not exceed two. The block method for k=2, with two off grid collocation point expressed in the form

$$\rho(z)=\left| z\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix} - \begin{bmatrix} 0&0&0&1&0&0&0&\frac{h}{2}\\ 0&0&0&1&0&0&0&h\\ 0&0&0&1&0&0&0&\frac{3h}{2}\\ 0&0&0&1&0&0&0&2h\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1 \end{bmatrix} \right| = z^{6}\left(z-1\right)^{2}$$

$$\rho(z) = z^{6}\left(z-1\right)^{2} = 0,$$

Hence, our method is zero-stable.

3.3 Consistency and convergence of implicit two-step hybrid block third derivatine method

The implicit two-step hybrid blcok third derivative method is consistent since it has order $p = 10 \geq 1$

and it is also convergent by consequence of Dahlquist theorem stated below

Theorem (Dahlquist): The necessary and sufficient conditions that a continuous LMM be convergent are that it be consistent and zero- stable.

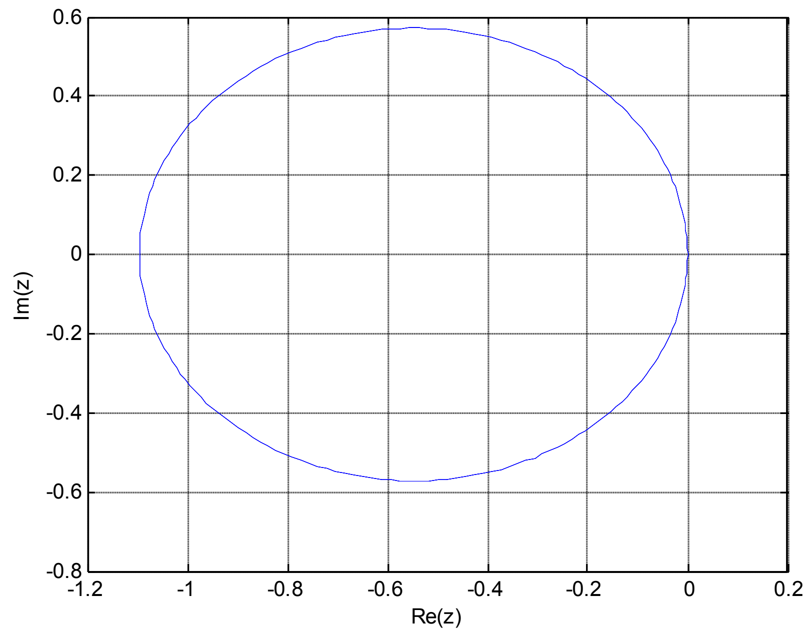
3.4 Regions of Absolute Stability (RAS)

Using MATLAB package, we were able to plot the stability region of the block methods (see figs below).

The stability polynomial for K=2 with two offstep point using Scientific Workplace software package we have

$$\begin{aligned}\bar{h}(w) = & -h^{12}\left(\left(\frac{79}{38320128000}\right)w^{3}-\left(\frac{79}{38320128000}\right)w^{4}\right)-h^{11}\left(\left(\frac{21367}{2414168064000}\right)w^{4}+\left(\frac{1466429}{16899176448000}\right)w^{3}\right)\\ & -h^{10}\left(\left(\frac{2000347}{30418517606400}\right)w^{4}+\left(\frac{49860871}{30418517606400}\right)w^{3}\right)+h^{9}\left(\left(\frac{1104277}{1086375628800}\right)w^{4}-\left(\frac{8591857}{434550251520}\right)w^{3}\right)\\ & -h^{8}\left(\left(\frac{133811}{120708403200}\right)w^{4}+\left(\frac{8376149}{34488115200}\right)w^{3}\right)-h^{7}\left(\left(\frac{11270689}{181062604800}\right)w^{4}+\left(\frac{388853}{745113600}\right)w^{3}\right)\\ & -h^{6}\left(\left(\frac{4113492623}{289700167680}\right)w^{3}-\left(\frac{145766287}{289700167680}\right)w^{4}\right)-h^{5}\left(\left(\frac{82801}{167650560}\right)w^{4}-\left(\frac{2482013}{301771008}\right)w^{3}\right)\\ & -h^{4}\left(\left(\frac{63941}{3991680}\right)w^{4}+\left(\frac{25126547}{83825280}\right)w^{3}\right)+h^{3}\left(\left(\frac{100}{891}\right)w^{4}+\left(\frac{122}{891}\right)w^{3}\right)-h^{2}\left(\left(\frac{16399}{57024}\right)w^{4}+\left(\frac{97469}{57024}\right)w^{3}\right)+w^{4}-w^{3}\end{aligned}$$

Using MATLAB software, the absolute stability region of the new method is plotted and shown below.



3.5 Numerical example

Problem I. We consider a highly stiff problem

$y''+1001y'+1000y, \quad y(0)=1, y'(0)=-1$

Exact Solution: $y(x)=\exp(-x)\ h=\frac{1}{10}$

Table 1. Comparison of the proposed method with

x-values	Exact solution	Computed solution	Error in our method	Error in [10]
0.100	0.90483741803595957316	0.90483741803595957235	8.1000E(-19)	1.054712E(-14)
0.200	0.81873075307798185867	0.81873075307798187177	1.3100E(-17)	1.776357E(-14)
0.300	0.74081822068171786607	0.74081822068171787792	1.1850E(-17)	2.342571E(-14)
0.400	0.67032004603563930074	0.67032004603563931143	1.0690E(-17)	2.797762E(-14)
0.500	0.60653065971263342360	0.60653065971263343325	9.6500E(-18)	3.130829E(-14)
0.600	0.54881163609402643263	0.54881163609402644133	8.7000E(-18)	3.397282E(-14)
0.700	0.49658530379140951470	0.49658530379140952257	7.8700E(-18)	3.563816E(-14)
0.800	0.44932896411722159143	0.44932896411722159850	7.0700E(-18)	3.674838E(-14)
0.900	0.40656965974059911188	0.40656965974059911826	6.3800E(-18)	3.730349E(-14)
1.00	0.36787944117144232160	0.36787944117144232733	5.7300E(-18)	3.741452E(-14)

Problem II. $f(x,y,y')=y', \quad y(0)=1,\ y'(0)=\frac{1}{2}\ ,\ 0\le x\le 1.$

Exact Solution: $y(x)=1-e^{x}\ \ with\ h=\frac{1}{100}$

Table 2.

x-values	Exact solution	Computed solution	Error in our method	Error in [13]
0.100	-0.10517091807564762480	-0.10517091807564762481	1.000E(-20)	8.326679(-17)
0.200	-0.22140275816016983390	-0.22140275816016983392	2.000E(-20)	2.775557(-16)
0.300	-0.34985880757600310400	-0.34985880757600310398	2.000E(-20)	5.551115(-16)
0.400	-0.49182469764127031780	-0.49182469764127031782	2.000E(-20)	9.436896(-16)
0.500	-0.64872127070012814680	-0.64872127070012814684	4.000E(-20)	2.109424(-15)
0.600	-0.82211880039050897490	-0.82211880039050897486	4.000E(-19)	3.219647(-15)
0.700	-1.01375270747047652160	-1.01375270747047652160	0.000E(00)	4.440892(-15)
0.800	-1.22554092849246760460	-1.22554092849246760460	0.000E(00)	5.995204(-15)
0.900	-1.45960311115694966380	-1.45960311115694966380	0.000E(00)	7.771561(-15)
1.00	-1.71828182845904523540	-1.71828182845904523530	1.000E(-19)	1.065814(-14)

4 Conclusions

It is evident from the above tables that our proposed methods are indeed accurate, and can handle stiff equations. Also in terms of stability analysis, the method is $A-stable$.

Comparing the new method with the existing method [7] and [13], the result presented in the Tables 1 and 2 shows that the new method performs better than the existing method [7] and [13] since the order of new method is higher than that of the method compared with, it is expected for the accuracy to be better . In this article, a two-step block method with two off-step points is derived via the interpolation and collocation approach. The developed method is converges, with a region of absolute stability and order Ten.

Competing Interests

Authors have declared that no competing interests exist.

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